# Spinning and Rolling of an Unbalanced Disk

## Master's Thesis

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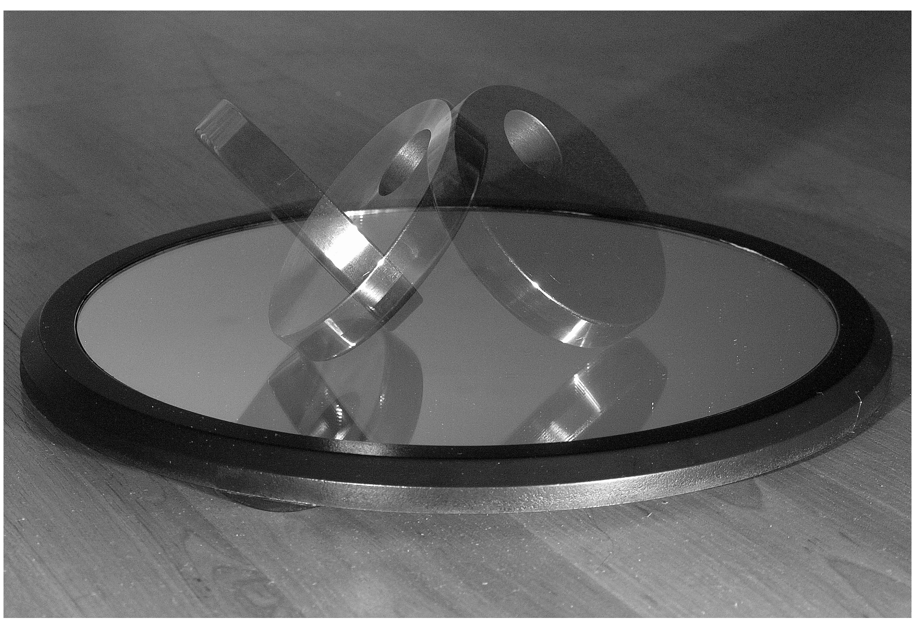
**Abstract**

The spinning and rolling motion of a circular disk under the influence of gravity is investigated. The mass distribution of the disk is unbalanced; that is, its centre of mass does not coincide with the geometric centre of the disk. This non-integrable, nonholonomic system combines the motion of a symmetric disk with the rolling of an unbalanced spherical ball. The Hamilton-Pontryagin variational approach to nonholonomic systems is used to formulate the equations of motion. The solutions of these equations are investigated in a variety of situations.



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# 1 Introduction

The work for this project was inspired by seeing the motion of the physics toy called Euler's disk [11]. This solid metal disk exhibits a similar type of motion to when you drop a coin, except it lasts for 2 minutes! As the disk slowly loses energy it leans more towards the horizontal, rolls slower and passes from rolling to "wobbling" at a definite frequency. As the disk leans closer to the flat, it wobbles faster. That is, the frequency of the wobbling and the phase speed of its point of contact increases dramatically, although the disk itself is hardly moving. The motion ends in a climax as the vibration frequency of the wobbling increases to a high rate and then suddenly stops.

Further inspiration was obtained from a ring which had a cutout, so consequently its mass was unbalanced. When that was spun, as the ring approached the flat it seemed to draw out a spiral and exhibited other motions not seen in Euler's disk. I had to find out the mathematics behind it! The ring with a cutout is an example of a spinning disk with an unbalanced mass distribution. To this end we purchased an Euler's disk and took it to a machine shop to have an off-center hole put in it, hence making it unbalanced. An image of this disk can be seen on the front cover.

The problem of the symmetric spinning disk has been studied in many papers (such as [2, 4, 7, 17, 9]) although all except one [9] use Euler angles, which leads to a lot of messy algebra. We follow the Geometric mechanics based approach outlined in [9]. This provides an elegant approach that allows us to write down the equations of motion in a single line. There are also close links between the unbalanced disk and an unbalanced ball rolling on the plane, as studied by Chaplygin and others[15, 9, 8]. The unbalanced disk provides an excellent example of a nonholonomic system and we include a brief overview concerning some of the implications of a system being nonholonomic.

In this report, we

- derive the equation of motion for the unbalanced disk
- compare equations with those for the unbalanced ball and the balanced disk
- investigate 2D dynamics of the unbalanced disk in the vertical plane
- investigate rolling on a curved surface
- examine 3D motions of the unbalanced disk with constant potential energy
- look at the motion of spinning cylinders

The main content of this report is

- Section 2 formulates the equations of motion for nonholonomically constrained motion using Hamilton-Pontryagin variational principle in the framework of Geometric Mechanics.
- Section 3 describes conservation of energy for rolling without slipping of the unbalanced disk.
- Section 4 compares the unbalanced disk to the rolling ball and balanced Euler disk.

- Section 5 discusses the effects of nonholonomic constraints on conservation laws and Noether's theorem.
- Section 6 investigates vertical rolling. This includes rolling on a curved surface and coupled rollers.
- Section 7 constrains the rolling of an unbalanced disk to have constant potential energy. This includes using penalty to implement the constraint.
- Section 8 deduces how the wobbling frequency of the symmetric disk increases as the angle of the disk to the horizontal decreases.
- Section 9 expands the model to spinning cylinders.
- Section 10 discusses the numerical integration of the equations of motion.

## 2 Formulating Equations of Motion

The formulation of the equations of motion follows the method used in Geometric Mechanics II: Rotating, Translating and Rolling[9], which derives equations of motion for a rolling ball and a symmetric disk.

We consider a flat, heavy disk of mass $m$ and radius $r$ rolling on a horizontal surface without slipping. The disk we consider has an unbalanced mass distribution, that is, its center of mass does not necessarily coincide with the center of the disk. We choose a right-handed system of unit vectors $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ which represent the reference frame of the disk. These are chosen such that the origin of the axis lies at the centre of symmetry of the disk, $\boldsymbol{E}_3$ is perpendicular to the disk and the disk lies in the $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ plane. It is easiest to choose the $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ so that the inertia tensor $\mathbb{I}$ becomes diagonal in these coordinates. This helps to simplify the equations of motion.

We also need a set of spatial vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. These are chosen such that $\boldsymbol{e}_3$ points upwards perpendicular to the horizontal surface. Since we are considering motion under gravity, this force will be in the negative $\boldsymbol{e}_3$ direction, pulling the disk down towards the plane.

In most cases it is convenient to choose the reference frame to coincide with the spatial frame, that is, when the disk lies flat. However, in certain cases, like in the section on 2D rolling, it is easier to choose the reference frame to be when the disk is on it's side.

The motion of the disk can be expressed in terms of the position and orientation of the disk. This corresponds to the action of $SE(3) \cong SO(3) \circledS \mathbb{R}^3$. $SO(3)$ consists of rotations and defines the orientation of the disk. The elements of $\mathbb{R}^3$ represent translations to determine the position of the disk in space. $\circledS$ represents a semi-direct product.

**Remark** A semi-direct product group $G = H \circledS N$ is such that $N$ is a normal subgroup, $H$ is a subgroup and any element of $G$ can be expressed uniquely as the product of an element $N$ and an element of $H$[31]. In this case, $\mathbb{R}^3$ is the normal subgroup of $SE(3)$. An element of $SE(3)$ can be written as $(R, v)$ where $R \in SO(3)$ and $v \in \mathbb{R}^3$. This is similar to the direct product of groups, however, with a semi-direct product, the group multiplication is different:

$$(R, v)(\tilde{R}, \tilde{v}) = (R\tilde{R}, R\tilde{v} + v)$$

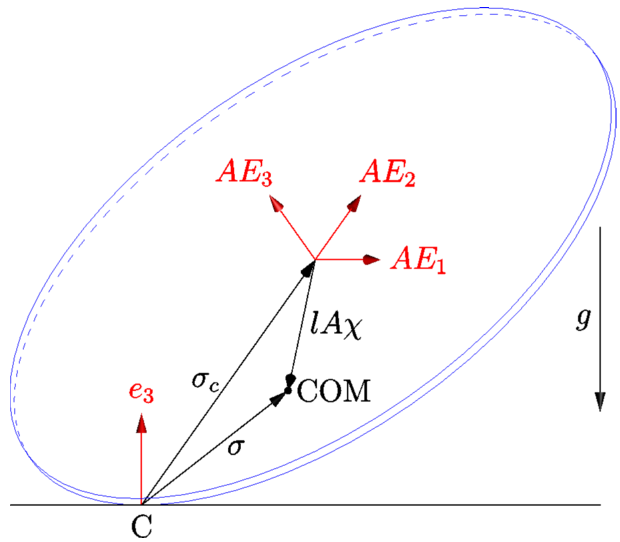



***Figure 2.1:*** *A circular disk is spinning and rolling under the action of gravity.* $\boldsymbol{\sigma}$ *is the spatial vector from the point of contact to the center of mass.* $\boldsymbol{\sigma}_c$ *is the spatial vector from the point of contact to the center of the disk.* $lA\boldsymbol{\chi}$ *is the spatial vector from the center of the disk to the center of mass.*

The inverse is then defined by

$$(R, v)^{-1} = (R^{-1}, -R^{-1}v)$$

Recall that a subgroup $N$ of $G$ is normal if for any $n \in N$ and any $g \in G$ then $gng^{-1} \in N$ [30]. $\mathbb{R}^3$ as a subgroup of $SE(3)$ is represented by elements of the form $(I, u) \in SE(3)$ where $I$ is the $3 \times 3$ identity matrix. Then normality of $\mathbb{R}^3 \subset SE(3)$ is easy to check using the definition:

$$\begin{aligned}(R, v)(I, u)(R, v)^{-1} &= (R, v)(I, u)(R^{-1}, -R^{-1}v)\\ &= (RIR^{-1}, v + Ru - RIR^{-1}v)\\ &= (I, Ru) \in \mathbb{R}^3 \subset SE(3)\end{aligned}$$

We recognise here the AD-operation of Lie groups $AD : G \times G \to G$. So that any normal subgroup is closed under the AD operation $AD : G \times N \to N$. For more information on adjoint actions see [9]. For a more detailed description of semi-direct product Lie algebras with applications see [10], [16] and references therein.

□

The orientation of the disk at time $t$ relative to the reference configuration is given by $(A(t)\boldsymbol{E}_1, A(t)\boldsymbol{E}_2, A(t)\boldsymbol{E}_3)$ where $A = A(t) \in SO(3)$. Now that we have our coordinate systems set up, we need to define some key vectors (see figure 2.1):

- $\boldsymbol{\chi}$ will be the body unit vector pointing from the center of symmetry to the center of mass. The center of mass is a distance $l$ from the center of symmetry.
- $\boldsymbol{\sigma}(t)$ will be the spatial vector from the point of contact $C$ to the centre of mass.

- $\boldsymbol{\sigma}_c(t)$ will be the spatial vector from $C$ to the centre of symmetry of the disk. This leads to the relation:
$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_c(t) + lA(t)\boldsymbol{\chi}$$
Note we have $A(t)\boldsymbol{\chi}$ since $\boldsymbol{\chi}$ is in the body and $A(t)$ takes us into the spatial frame.

- $\boldsymbol{s}(t)$ will be the vector in the body frame corresponding to $\boldsymbol{\sigma}(t)$ in the spatial frame. That is, $\boldsymbol{s}(t)$ is the body vector from the point of contact to the centre of symmetry. We have a similar definition for $\boldsymbol{s}_c(t)$, from $C$ to the center of symmetry.

- Since the orientation matrix $A(t)$ moves us between the spatial and body frames we get the relations:
$$\boldsymbol{\sigma}(t) = A(t)\boldsymbol{s}(t) \implies \boldsymbol{s}(t) = A^{-1}(t)\boldsymbol{\sigma}(t) = \boldsymbol{s}_c(t) + l\boldsymbol{\chi}$$
$$\boldsymbol{\sigma}_c(t) = A(t)\boldsymbol{s}_c(t) \implies \boldsymbol{s}_c(t) = A^{-1}(t)\boldsymbol{\sigma}_c(t)$$

**Remark** I will often switch between vector notation (in bold) and non-vector notation. This is just for ease of notation, especially when it comes to group actions and taking variations in the Hamilton-Pontryagin principle. In some cases this is a trivial mapping: for vectors $s \leftrightarrow \boldsymbol{s}$ but in some cases this is less trivial. This less trivial mapping is for elements of the Lie Algebra $\mathfrak{so}(3)$. These are anti-symmetric matrices and so can be expressed in vector notation via the hat-map:

$$\Omega = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \quad \Longleftrightarrow \quad \boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$$

then the product of $\Omega \in \mathfrak{so}(3)$ and $s \in \mathbb{R}^3$ maps over by

$$\Omega s \quad \Longleftrightarrow \quad \boldsymbol{\Omega} \times \boldsymbol{s}$$

This is called the hat-map because the matrix form of elements of the Lie Algebra are often denoted with a hat (such as $\hat{\Omega}$). Since $\mathfrak{so}(3)$ is a Lie algebra we can define the adjoint action $\mathrm{ad} : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3)$. This is the Lie Poisson bracket which defines the product structure on the vector space of anti-symmetric matrices. In the case of a matrix Lie algebra it is simply the matrix commutator. This commutator maps to a cross product under the hat-map.

$$\mathrm{ad}_\Omega\, \Xi = [\Omega, \Xi] = \Omega\Xi - \Xi\Omega \quad \Longleftrightarrow \quad \mathrm{ad}_{\boldsymbol{\Omega}}\, \boldsymbol{\Xi} = \boldsymbol{\Omega} \times \boldsymbol{\Xi}$$

□

## 2.1 Rolling Constraint

What do we mean by rolling? Rolling of the disk means that the point of contact of the disk to the plane is not sliding, hence it can also be referred to as a no-slip condition. An equivalent definition is that the motion of any point in the disk must be obtained by a rotation about the point of contact. It must be if the point of contact is not sliding. The only point in the disk that is significant to us is the centre of mass, so we concentrate on that. In the spatial frame, the angular frequency is $\omega(t) = \dot{A}A^{-1}(t) \in \mathfrak{so}(3)$.

Let $\boldsymbol{x}(t)$ be the spatial vector pointing from the origin to the centre of mass, then the rolling constraint says that the velocity of the centre of mass is given by a rotation of the spatial vector $\boldsymbol{\sigma}(t)$. So the rolling constraint can be written as:

$$\dot{\boldsymbol{x}}(t) = \omega(t)\boldsymbol{\sigma}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{\sigma}(t) \tag{2.1}$$

Note the use of the hat-map. This is the constraint in the spatial frame. It is useful to transform this into the body frame. Let $\boldsymbol{Y} = A^{-1}\dot{\boldsymbol{x}}(t)$ be the corresponding vector in the body. Then the rolling constraint in the body can be written as:

$$\boldsymbol{Y} = A^{-1}\dot{A}A^{-1}\boldsymbol{\sigma}(t) = A^{-1}\dot{A}\boldsymbol{s} = \Omega\boldsymbol{s} = \boldsymbol{\Omega} \times \boldsymbol{s} \tag{2.2}$$

where $\Omega = A^{-1}\dot{A} \in \mathfrak{so}(3)$ is the angular frequency in the body.

**Remark** This rolling constraint is known as a nonholonomic constraint. In simple terms this means that it cannot be expressed in the form $df = 0$ for some function $f$ [5]. An example of a holonomic constraint is $|x|^2 = 1$ since this can be expressed as $d|x|^2 = 0$. Nonholonomic constraints are trickier to handle than holonomic constraints. We shall discuss this in more detail later.

□

## 2.2 Lagrangian

The Lagrangian is the kinetic energy $T$ minus the potential energy $V$. The kinetic energy (KE) consists of 2 parts, the rotational KE and the translational KE:

$$T = \underbrace{\frac{1}{2}\langle\Omega, \mathbb{I}\Omega\rangle}_{\text{rotational}} + \underbrace{\frac{m}{2}|\dot{x}|^2}_{\text{kinetic}} = \frac{1}{2}\langle\Omega, \mathbb{I}\Omega\rangle + \frac{m}{2}|Y|^2 \tag{2.3}$$

The potential energy is due to gravity and depends on the height of the centre of mass $z(t)$ above the plane on which the disk is spinning or rolling. The potential energy is therefore given by $V = mgz(t)$, where

$$z(t) = \langle\sigma(t), \boldsymbol{e}_3\rangle = \langle As(t), \boldsymbol{e}_3\rangle = \left\langle s, A^{-1}\boldsymbol{e}_3\right\rangle = \langle s, \Gamma\rangle \tag{2.4}$$

where $\Gamma = A^{-1}\boldsymbol{e}_3$ is the vertical spatial axis as seen in the body. Since $\boldsymbol{e}_3$ is a unit vector and $A$ is orthogonal, $|\Gamma|^2 = |\boldsymbol{e}_3|^2 = 1$.

We can now write down the Lagrangian for the system:

$$\mathcal{L}(\Omega, Y, \Gamma) = \frac{1}{2}\langle\Omega, \mathbb{I}\Omega\rangle + \frac{m}{2}|Y|^2 - mg\langle s(\Gamma), \Gamma\rangle \tag{2.5}$$

In vector notation, obtained via the hat-map, the Lagrangian is

$$\mathcal{L}(\boldsymbol{\Omega}, \boldsymbol{Y}, \boldsymbol{\Gamma}) = \frac{1}{2}\boldsymbol{\Omega}\cdot\mathbb{I}\boldsymbol{\Omega} + \frac{m}{2}|\boldsymbol{Y}|^2 - mg\boldsymbol{s}(\boldsymbol{\Gamma})\cdot\boldsymbol{\Gamma} \tag{2.6}$$

## 2.3 Some Extra Ingredients

It will help us later on if we have some extra tools/expressions to help with our calculations. First we notice (see figure 2.1) that $\boldsymbol{\sigma}_c$ is perpendicular to both $A\boldsymbol{E}_3$ and the vector $\boldsymbol{e}_3 \times A\boldsymbol{E}_3$[9]. $\boldsymbol{\sigma}_c$ has length $r$ so we can write

$$\boldsymbol{\sigma}_c = r\frac{A\boldsymbol{E}_3 \times (\boldsymbol{e}_3 \times A\boldsymbol{E}_3)}{|A\boldsymbol{E}_3 \times (\boldsymbol{e}_3 \times A\boldsymbol{E}_3|} = r\frac{A\boldsymbol{E}_3 \times (\boldsymbol{e}_3 \times A\boldsymbol{E}_3)}{\sqrt{1-(\boldsymbol{e}_3 \cdot A\boldsymbol{E}_3)^2}} \tag{2.7}$$

Transforming this into the body (to get $\boldsymbol{s}_c$) and using the fact that the rotation of a cross product is the cross product of the rotated vectors we get:

$$\begin{aligned}\boldsymbol{s}_c = A^{-1}\boldsymbol{\sigma}_c &= \frac{r}{\sqrt{1-(\boldsymbol{e}_3 \cdot A\boldsymbol{E}_3)^2}}\,\boldsymbol{E}_3 \times A^{-1}(\boldsymbol{e}_3 \times A\boldsymbol{E}_3)\\ &= \frac{r}{\sqrt{1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2}}\,\boldsymbol{E}_3 \times (A^{-1}\boldsymbol{e}_3 \times \boldsymbol{E}_3)\\ &= \frac{r}{\sqrt{1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2}}\,\boldsymbol{E}_3 \times (\boldsymbol{\Gamma} \times \boldsymbol{E}_3)\\ &= \frac{r}{\sqrt{1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2}}\big(\boldsymbol{\Gamma} - (\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)\boldsymbol{E}_3\big)\end{aligned} \tag{2.8}$$

It is then some simple algebra to calculate $\dot{\boldsymbol{s}}$:

$$\dot{\boldsymbol{s}} = \dot{\boldsymbol{s}}_c = \frac{r}{\sqrt{1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2}}\dot{\boldsymbol{\Gamma}} - \frac{r}{\big(1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2\big)^3}\big(\dot{\boldsymbol{\Gamma}} \cdot \boldsymbol{E}_3\big)\big(\boldsymbol{E}_3 - (\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)\boldsymbol{\Gamma}\big) \tag{2.9}$$

where we have used the fact that $\boldsymbol{\chi}$ is constant. Here we notice that $\dot{\boldsymbol{s}} \cdot \boldsymbol{E}_3 = 0$.

In some cases we need to calculate derivatives like $\frac{\partial f(\boldsymbol{s})}{\partial \boldsymbol{\Gamma}}$ (for instance, the derivative of the Lagrangian). For this we need to use the chain rule:

$$\frac{\partial f(\boldsymbol{s})}{\partial \boldsymbol{\Gamma}} = \frac{\partial f(\boldsymbol{s})}{\partial s_j}\frac{\partial s_j}{\partial \boldsymbol{\Gamma}}$$

So it makes sense to calculate the following

$$\eta_j\frac{\partial s_j}{\partial \boldsymbol{\Gamma}} = \frac{r}{\sqrt{1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2}}\boldsymbol{\eta} + \frac{(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)(\boldsymbol{\eta} \cdot \boldsymbol{\Gamma})\boldsymbol{E}_3 - (\boldsymbol{\eta} \cdot \boldsymbol{E}_3)\boldsymbol{E}_3}{\big(1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2\big)^{3/2}} \tag{2.10}$$

This equation gives us the relation $\boldsymbol{\Gamma} \cdot \delta\boldsymbol{s} = 0$ since

$$\boldsymbol{\Gamma} \cdot \delta\boldsymbol{s} = \Gamma_j\frac{\partial s_j}{\partial \boldsymbol{\Gamma}} \cdot \delta\boldsymbol{\Gamma} = \frac{r}{\sqrt{1-(\boldsymbol{\Gamma} \cdot \boldsymbol{E}_3)^2}}\boldsymbol{\Gamma} \cdot \delta\boldsymbol{\Gamma} = 0 \tag{2.11}$$

where we use the fact that $|\boldsymbol{\Gamma}|^2 = 1$ which implies that $\delta|\boldsymbol{\Gamma}|^2 = 2\boldsymbol{\Gamma} \cdot \delta\boldsymbol{\Gamma} = 0$. The relation $\boldsymbol{\Gamma} \cdot \delta\boldsymbol{s} = 0$ will be very useful when it comes to taking derivatives of the Lagrangian.

## 2.4 Variational Principle

We now have the reduced Lagrangian and the rolling constraint distribution written in body coordinates. The standard Hamilton's principle used in Lagrangian mechanics is not applicable to nonholonomic systems. Therefore, to get the equations of motion, we apply the Hamilton-Pontryagin variational principle which was first outlined in [3] and later applied to rolling balls and symmetric disks in [9]. You may notice that Marsden and Bou-Rabee's paper [3] was in fact published after the release of Prof. Darryl Holm's book: [9]. This is due to Prof. Holm's close contact with Jerrold Marsden and Nawaf Bou-Rabee. Prof. Holm was able to see and implement these methods in his book before the official publishing of the Hamilton-Pontryagin paper.

We need to be careful with how we treat the nonholonomic constraint. With this kind of constraint we do not get the correct answer if we substitute in the constraint before taking variations. To get the correct answer we must be careful to take variations which are restricted to the constraint distribution $\mathcal{D}$ defined by the rolling constraint:

$$\mathcal{D} = \big\{ (\boldsymbol{\Omega}, \boldsymbol{Y}, \boldsymbol{\Gamma}) \;\big|\; \boldsymbol{Y} = \boldsymbol{\Omega} \times \boldsymbol{s}(\boldsymbol{\Gamma}) \big\} \tag{2.12}$$

We write down the left-trivialised action to be

$$\mathcal{S} = \int_a^b \mathcal{L}(\Omega, Y, \Gamma) + \big\langle \Pi, A^{-1}\dot{A} - \Omega \big\rangle + \big\langle \kappa, A^{-1}\boldsymbol{e}_3 - \Gamma \big\rangle + \big\langle \lambda, A^{-1}\dot{x} - Y \big\rangle \; dt \tag{2.13}$$

Then the equations of motion arise from the left-trivialised Hamilton-Pontryagin principle[3]

$$0 = \delta\mathcal{S} = \delta \int_a^b \mathcal{L}(\Omega, Y, \Gamma) + \big\langle \Pi, A^{-1}\dot{A} - \Omega \big\rangle + \big\langle \kappa, A^{-1}\boldsymbol{e}_3 - \Gamma \big\rangle + \big\langle \lambda, A^{-1}\dot{x} - Y \big\rangle \; dt \tag{2.14}$$

## 2.5 Calculating Variations

Before we can take variations of the action defined above we need to calculate the variations of each of the variables in the Lagrangian.

$$\delta\Omega = \delta(A^{-1}\dot{A}) = A^{-1}\delta\dot{A} - A^{-1}\delta A A^{-1}\dot{A} \tag{2.15}$$

We then define the independent variation $\eta = A^{-1}\delta A \in \mathfrak{so}(3)$ to give

$$\begin{aligned} \dot{\eta} &= A^{-1}\delta\dot{A} - A^{-1}\dot{A}A^{-1}\delta A \\ \implies \delta\Omega &= \delta(A^{-1}\dot{A}) = \dot{\eta} + \mathrm{ad}_\Omega\, \eta \end{aligned} \tag{2.16}$$

where the adjoint action $\mathrm{ad} : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3)$ is the matrix commutator as defined earlier.

$$\delta\Gamma = \delta(A^{-1}\boldsymbol{e}_3) = -A^{-1}\delta A A^{-1}\boldsymbol{e}_3 = -\eta\Gamma \tag{2.17}$$

We now look at the variations of $Y$. This is where we must be careful since $Y$ is constrained by $Y = \Omega s$. We must take variations which stay within the constraint distribution. We use the relation:

$$Y = A^{-1}\dot{x} = A^{-1}\dot{A}s \quad \implies \quad A^{-1}\delta x = (A^{-1}\delta A)s = \eta s \tag{2.18}$$

so that

$$\begin{aligned}\delta Y = \delta(A^{-1}\dot{x}) &= A^{-1}\delta\dot{x} - \eta A^{-1}\dot{x} \\ &= \frac{d}{dt}(A^{-1}\delta x) - \frac{d}{dt}(A^{-1})\delta x - \eta\Omega s \\ &= \frac{d}{dt}(\eta s) + \Omega\eta s - \eta\Omega s \\ &= \frac{d}{dt}(\eta s) + (\mathrm{ad}_\Omega\,\eta)s \qquad (2.19) \\ & \qquad (2.20)\end{aligned}$$

Before we can calculate variations of the action functional we must first define some operators. We define the coadjoint operator $\mathrm{ad}^* : \mathfrak{so}(3) \times \mathfrak{so}(3)^* \to \mathfrak{so}(3)^*$ where $\mathfrak{so}(3)^*$ is the dual to the Lie algebra $\mathfrak{so}(3)$. We define it using the pairing[9]:

$$\langle \mathrm{ad}^*_\Omega\,\mu, \eta\rangle = \langle \mu, \mathrm{ad}_\Omega\,\eta\rangle \qquad (2.21)$$

We also define the diamond operator, $\diamond$, using the pairing[9]:

$$\langle \kappa \diamond \Gamma, \eta\rangle = \langle \kappa, -\eta\Gamma\rangle \qquad (2.22)$$

We now have all we need to take variations of the action functional and obtain our equations of motion:

$$\begin{aligned}0 = \delta\mathcal{S} = \int_a^b &\left\langle \frac{\partial\mathcal{L}}{\partial\Omega} - \Pi, \delta\Omega\right\rangle + \left\langle \frac{\partial\mathcal{L}}{\partial\Gamma} - \kappa, \delta\Gamma\right\rangle + \left\langle \frac{\partial\mathcal{L}}{\partial Y} - \lambda, \delta Y\right\rangle \\ &+ \left\langle \Pi, \delta(A^{-1}\dot{A})\right\rangle + \left\langle \kappa, \delta(A^{-1}\boldsymbol{e}_3)\right\rangle + \left\langle \lambda, \delta(A^{-1}\dot{x})\right\rangle dt \qquad (2.23)\end{aligned}$$

The first three terms define the Lagrange multipliers $\Pi, \kappa, \lambda$ whilst, after some manipulation, the final three terms will give the equations of motion:

$$\begin{aligned}&\int_a^b \left\langle \Pi, \delta(A^{-1}\dot{A})\right\rangle + \left\langle \kappa, \delta(A^{-1}\boldsymbol{e}_3)\right\rangle + \left\langle \lambda, \delta(A^{-1}\dot{x})\right\rangle dt \\ &= \int_a^b \langle \Pi, \dot{\eta} + \mathrm{ad}_\Omega\,\eta\rangle + \langle \kappa, -\eta\Gamma\rangle + \left\langle \lambda, \frac{d}{dt}(\eta s) + (\mathrm{ad}_\Omega\,\eta)s\right\rangle dt \\ &= \int_a^b \left\langle -\dot{\Pi} + \mathrm{ad}^*_\Omega\,\Pi + \kappa\diamond\Gamma + \dot{\lambda}\diamond s - \mathrm{ad}^*_\Omega(\lambda\diamond s), \eta\right\rangle dt + \Big[\langle \Pi - \lambda\diamond s, \eta\rangle\Big]_a^b \\ &= -\int_a^b \left\langle \left(\frac{d}{dt} - \mathrm{ad}^*_\Omega\right)(\Pi - \lambda\diamond s) - \kappa\diamond\Gamma + \lambda\diamond\dot{s}, \eta\right\rangle dt + \Big[\langle \Pi - \lambda\diamond s, \eta\rangle\Big]_a^b\end{aligned}$$

We assume the variations vanish at the endpoints so that we obtain the equations of motion:

$$\left(\frac{d}{dt} - \mathrm{ad}^*_\Omega\right)(\Pi - \lambda\diamond s) = \kappa\diamond\Gamma - \lambda\diamond\dot{s} \qquad (2.24)$$

where

$$\Pi = \frac{\partial\mathcal{L}}{\partial\Omega}, \qquad \kappa = \frac{\partial\mathcal{L}}{\partial\Gamma}, \qquad \lambda = \frac{\partial\mathcal{L}}{\partial Y} \qquad (2.25)$$

We also have an auxiliary equation for the evolution of $\mathbf{\Gamma}$. This can be computed directly:

$$\dot{\mathbf{\Gamma}} = \frac{d}{dt} A^{-1} \boldsymbol{e}_3 = -A^{-1} \dot{A} A^{-1} \boldsymbol{e}_3 = -\Omega \mathbf{\Gamma} = -\mathbf{\Omega} \times \mathbf{\Gamma} \tag{2.26}$$

It is easiest to work in vector form since we are very familiar with vectors. To this end we rewrite the equations of motion in vector form. We use that hat-map and recall that the adjoint operator becomes the matrix cross product so that the coadjoint operator $\mathrm{ad}^*$ becomes the negative cross product:

$$\begin{aligned} \langle \mathrm{ad}^*_{\mathbf{\Omega}} \mathbf{\Pi}, \boldsymbol{\eta} \rangle &= \langle \mathbf{\Pi}, \mathrm{ad}_{\mathbf{\Omega}} \boldsymbol{\eta} \rangle \\ &= \mathbf{\Pi} \cdot \mathbf{\Omega} \times \boldsymbol{\eta} \\ &= -\mathbf{\Omega} \times \mathbf{\Pi} \cdot \boldsymbol{\eta} \\ &= \langle -\mathbf{\Omega} \times \mathbf{\Pi}, \boldsymbol{\eta} \rangle \end{aligned}$$

The diamond operator also becomes the cross product since, via the hat-map, $\eta\Gamma \to \boldsymbol{\eta} \times \mathbf{\Gamma}$:

$$\begin{aligned} \langle s \diamond \Gamma, \eta \rangle &= \langle s, -\eta \Gamma \rangle \\ &= -\boldsymbol{s} \cdot \boldsymbol{\eta} \times \mathbf{\Gamma} \\ &= \boldsymbol{s} \times \mathbf{\Gamma} \cdot \boldsymbol{\eta} \\ &= \langle \boldsymbol{s} \times \mathbf{\Gamma}, \boldsymbol{\eta} \rangle \end{aligned}$$

Therefore, the equations of motion written in vector form are:

$$\left( \frac{d}{dt} + \mathbf{\Omega} \times \right) (\mathbf{\Pi} - \boldsymbol{\lambda} \times \boldsymbol{s}) = \boldsymbol{\kappa} \times \mathbf{\Gamma} - \boldsymbol{\lambda} \times \dot{\boldsymbol{s}}$$
$$\dot{\mathbf{\Gamma}} = -\mathbf{\Omega} \times \mathbf{\Gamma}$$

Given the Lagrangian in vector form:

$$\mathcal{L}(\mathbf{\Omega}, \boldsymbol{Y}, \mathbf{\Gamma}) = \frac{1}{2} \mathbf{\Omega} \cdot \mathbb{I} \mathbf{\Omega} + \frac{m}{2} |\boldsymbol{Y}|^2 - mg \boldsymbol{s}(\mathbf{\Gamma}) \cdot \mathbf{\Gamma}$$

We can now compute $\mathbf{\Pi}, \boldsymbol{Y}, \mathbf{\Gamma}$ and use these to get the full equation of motion:

$$\mathbf{\Pi} = \frac{\partial \mathcal{L}}{\partial \mathbf{\Omega}} = \mathbb{I} \mathbf{\Omega}, \qquad \boldsymbol{\kappa} = \frac{\partial \mathcal{L}}{\partial \mathbf{\Gamma}} = -mg \boldsymbol{s}(\mathbf{\Gamma}), \qquad \boldsymbol{\lambda} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{Y}} = m \boldsymbol{Y}$$

$$\left( \frac{d}{dt} + \mathbf{\Omega} \times \right) \big( \mathbb{I} \mathbf{\Omega} - m (\mathbf{\Omega} \times \boldsymbol{s}) \times \boldsymbol{s} \big) = -mg \boldsymbol{s}(\mathbf{\Gamma}) \times \mathbf{\Gamma} - m (\mathbf{\Omega} \times \boldsymbol{s}) \times \dot{\boldsymbol{s}} \tag{2.27}$$

$$\dot{\mathbf{\Gamma}} = -\mathbf{\Omega} \times \mathbf{\Gamma}$$

where we have used the relation $\mathbf{\Gamma} \cdot \delta \boldsymbol{s} = 0$ in the calculation of $\partial \mathcal{L} / \partial \mathbf{\Gamma}$. Here we have also substituted in the rolling constraint $\boldsymbol{Y} = \mathbf{\Omega} \times \boldsymbol{s}$. This equation looks exactly the same as the equation for the symmetric disk in [9] but we must remember that our definition of $\boldsymbol{s}$ is different. In this equation $\boldsymbol{s}$ goes to the center of mass and not the center of symmetry (although these do coincide in the symmetric case).

## 2.6 Reconstructing the Full Solution

When we solve the system above we get $\mathbf{\Omega}(t)$ and $\mathbf{\Gamma}(t)$. We note that in essence we have reduced the problem as defined on $SE(3)$ to a problem on $\mathfrak{so}(3)$. However, due to the rolling constraint, given the orientation of the disk we can reconstruct the full motion of the disk in $SE(3) = SO(3)Ⓢ\mathbb{R}^3$. We have the reconstruction equation for $A$:

$$\dot{A} = A\Omega \tag{2.28}$$

Which will give us $A(t) \in SO(3)$. To find the position of the point of contact $\boldsymbol{X}(t)$ we note that

$$\begin{aligned}\boldsymbol{X}(t) &= \boldsymbol{x}(t) - \boldsymbol{\sigma}(t)\\ &= \boldsymbol{x}(t) - A(t)\boldsymbol{s}(t)\end{aligned} \tag{2.29}$$

where $\boldsymbol{x}(t)$ is the position of the center of mass. The motion of the center of mass is determined by the rolling constraint

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{\omega} \times \boldsymbol{\sigma}\\ &= \omega\boldsymbol{\sigma}\\ &= \dot{A}A^{-1}A\boldsymbol{s}\\ &= \dot{A}\boldsymbol{s}\end{aligned} \tag{2.30}$$

It is now simple to write down the motion equation for the point of contact:

$$\begin{aligned}\dot{\boldsymbol{X}} &= \dot{\boldsymbol{x}} - \dot{A}\boldsymbol{s} - A\dot{\boldsymbol{s}}\\ &= -A\dot{\boldsymbol{s}}\end{aligned} \tag{2.31}$$

Obviously, the point of contact must stay in the plane so we perform a quick check:

$$\begin{aligned}\dot{\boldsymbol{X}} \cdot \boldsymbol{e}_3 &= -A\dot{\boldsymbol{s}} \cdot \boldsymbol{e}_3\\ &= -\dot{\boldsymbol{s}} \cdot A^{-1}\boldsymbol{e}_3\\ &= -\dot{\boldsymbol{s}} \cdot \mathbf{\Gamma}\\ &= 0\end{aligned}$$

So the point of contact will never leave the plane, as required.

Given these equations we can now solve to find the full motion for a given set of initial conditions. Numerical integration of these equations will be discussed in section 10.

## 2.7 Constraint Before Variations

As stated earlier, one of the difficulties of nonholonomic constraints is that if we substitute in the constraint and then take variations we get a different answer than if we take variations and then substitute in the constraint. To really emphasise this point, in this section I will take variations after substituting in the constraint to see what we get.

Suppose we take the constrained Lagrangian:

$$\mathcal{L}_c(\Omega, \Gamma) = \frac{1}{2}\langle\Omega, \mathbb{I}\Omega\rangle + \frac{m}{2}|\Omega s(\Gamma)|^2 - mg\langle s(\Gamma), \Gamma\rangle \tag{2.32}$$

Then our Hamilton-Pontryagin action functional is

$$S = \int_a^b \mathcal{L}_c(\Omega, \Gamma) + \left\langle \Pi, A^{-1}\dot{A} - \Omega \right\rangle + \left\langle \kappa, A^{-1}\boldsymbol{e}_3 - \Gamma \right\rangle dt \tag{2.33}$$

Stationarity of this action functional gives the equations of motion:

$$\frac{d\Pi}{dt} - \mathrm{ad}^*_\Omega \Pi = \kappa \diamond \Gamma \tag{2.34}$$

where $\Pi = \partial\mathcal{L}_c/\partial\Omega$ and $\kappa = \partial\mathcal{L}_c/\partial\Gamma$.

If we now translate back into vector notation and calculate the derivatives we get:

$$\boldsymbol{\Pi} = \frac{\partial\mathcal{L}_c}{\partial\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{s} \times (\boldsymbol{\Omega} \times \boldsymbol{s}) \tag{2.35}$$

So the left hand side of the equation of motion remains the same. However, when we calculate the right hand side of the equation we get a big mess which is not the same as the equations of motion we previously calculated:

$$\boldsymbol{\kappa} = \frac{\partial\mathcal{L}_c}{\partial\boldsymbol{\Gamma}} = m\big[\boldsymbol{\Omega} \times (\boldsymbol{s} \times \boldsymbol{\Omega})\big]_i \frac{\partial s_i}{\partial\boldsymbol{\Gamma}} - mg\boldsymbol{s} \tag{2.36}$$

Making use of equation (2.10) and then taking the cross product of the above equation with $\boldsymbol{\Gamma}$ and we do not get the same equations as when we use the constraint after variations. This phenomenon can be likened to a property of standard calculus: We must take derivatives and then evaluate, not evaluate and then take derivatives!

**Remark** We can compare this to the planar pendulum written in polar coordinates. This has the constraint $\dot{r} = 0$. The unconstrained Lagrangian is

$$\mathcal{L}(r, \dot{r}, \theta, \dot{\theta}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

Which gives the equations of motion (from Hamilton's principle):

$$m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta, \quad \frac{d}{dt}mr^2\dot{\theta} = -mgr\sin\theta$$

Substituting in the constraint now (after taking variations) gives us the equation of motion

$$mr^2\ddot{\theta} = -mgr\sin\theta$$

If we now look at the Euler-Lagrange equations for the constrained Lagrangian:

$$\mathcal{L}_c(\theta, \dot{\theta}) = \frac{m}{2}r^2\dot{\theta}^2 + mgr\cos\theta$$

We get the exact same equation of motion. Therefore, this holonomic system is unaffected if we apply the constraint before or after taking variations.

□

### 2.8 Unconstrained Lagrangian

To emphasise the importance of staying on the constraint distribution when taking variations we can look at what happens when we don't! If we don't stay on the constraint distribution the variation of $Y$ changes:

$$\begin{aligned}\delta\left(A^{-1}\dot{x}\right) &= -A^{-1}\delta A A^{-1}\dot{x} + \frac{d}{dt}(A^{-1}\delta x) + A^{-1}\dot{A}A^{-1}\delta x \\ &= -\eta A^{-1}\dot{x} + \dot{\mu} + \Omega\mu \end{aligned} \tag{2.37}$$

where $\mu = A^{-1}\delta x$ is another variation independent of $\eta = A^{-1}\delta A$. This is the key difference. When we stick to the constraint distribution, $\mu$ is not independent of $\eta$. This mistake leads to the equations of motion:

$$\begin{aligned}\dot{\lambda} &= \lambda \diamond \Omega \\ \dot{\Gamma} &= \Gamma \diamond \Omega \\ \dot{\Pi} - \mathrm{ad}^*_{\Omega}\,\Pi - \lambda \diamond Y - \kappa \diamond \Gamma &= 0\end{aligned} \tag{2.38}$$

but $\lambda = \partial\mathcal{L}/\partial Y = mY$ so that $\lambda \diamond Y = 0$. This gives us a semi-decoupled system. We can solve for $\Omega$ and $\Gamma$ without knowing $Y$. If we try to implement the rolling constraint now, it will not work. Additionally, the auxiliary equation for $Y$ gives $|Y| = const$. This tells us that the translational kinetic energy is constant, which is not true in general.

## 3 Conservation of Energy

The Lagrangian has no explicit time dependence and therefore the disk has an energy which is conserved. This is given by

$$E(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) = \frac{1}{2}\left\langle \boldsymbol{\Omega}, \mathbb{I}\boldsymbol{\Omega}\right\rangle + \frac{m}{2}|\boldsymbol{\Omega} \times \boldsymbol{s}|^2 + mg\left\langle \boldsymbol{s}, \boldsymbol{\Gamma}\right\rangle \tag{3.1}$$

which is simply the sum of kinetic and potential energies. This can be expressed in the form of a Hamiltonian via the Legendre transformation of the constrained Lagrangian:

$$E(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) = \left\langle \frac{\partial \mathcal{L}_c}{\partial \boldsymbol{\Omega}}, \boldsymbol{\Omega}\right\rangle - \mathcal{L}_c(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) \tag{3.2}$$

where $\mathcal{L}_c$ is the constrained Lagrangian given by

$$\mathcal{L}_c(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) = \frac{1}{2}\left\langle \boldsymbol{\Omega}, \mathbb{I}\boldsymbol{\Omega}\right\rangle + \frac{m}{2}|\boldsymbol{\Omega} \times \boldsymbol{s}|^2 - mg\left\langle \boldsymbol{s}, \boldsymbol{\Gamma}\right\rangle \tag{3.3}$$

that is, the constrained Lagrangian is simply obtained by substituting in the rolling constraint given by $\boldsymbol{Y} = \boldsymbol{\Omega} \times \boldsymbol{s}$.

### 3.1 Proof of Conservation of Energy

The easiest way to prove the conservation of Energy for our system is to use the Hamiltonian form, which tells us

$$\frac{dE}{dt} = \left\langle \frac{d}{dt}\frac{\partial \mathcal{L}_c}{\partial \boldsymbol{\Omega}}, \boldsymbol{\Omega}\right\rangle - \left\langle \frac{\partial \mathcal{L}_c}{\partial \boldsymbol{\Gamma}}, \dot{\boldsymbol{\Gamma}}\right\rangle \tag{3.4}$$

So lets look at this component-wise:

$$\frac{d}{dt}\frac{\partial \mathcal{L}_c}{\partial \boldsymbol{\Omega}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} + \frac{\partial}{\partial \boldsymbol{\Omega}}\left(\frac{m}{2}\left|\boldsymbol{\Omega} \times \boldsymbol{s}\right|^2\right)\right) = \frac{d}{dt}\left(\boldsymbol{\Pi} - \boldsymbol{\lambda} \times \boldsymbol{s}\right) \tag{3.5}$$

To find this time derivative we use the motion equation given by the nonholonomic Hamilton-Pontryagin principle

$$\left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\left(\boldsymbol{\Pi} - \boldsymbol{\lambda} \times \boldsymbol{s}\right) = \boldsymbol{\kappa} \times \boldsymbol{\Gamma} - \boldsymbol{\lambda} \times \dot{\boldsymbol{s}} \tag{3.6}$$

So that when we take the inner product with $\boldsymbol{\Omega}$ we get

$$\left\langle \frac{d}{dt}\frac{\partial \mathcal{L}_c}{\partial \boldsymbol{\Omega}}, \boldsymbol{\Omega} \right\rangle = \langle \boldsymbol{\kappa} \times \boldsymbol{\Gamma}, \boldsymbol{\Omega}\rangle - \langle \boldsymbol{\lambda} \times \dot{\boldsymbol{s}}, \boldsymbol{\Omega}\rangle \tag{3.7}$$

The remaining term to look at is

$$\begin{aligned}\left\langle \frac{\partial \mathcal{L}_c}{\partial \boldsymbol{\Gamma}}, \dot{\boldsymbol{\Gamma}} \right\rangle &= \dot{\boldsymbol{\Gamma}} \cdot \frac{\partial}{\partial \boldsymbol{\Gamma}}\left(\frac{m}{2}\left|\boldsymbol{\Omega} \times \boldsymbol{s}\right|^2\right) - mg\boldsymbol{s} \cdot \dot{\boldsymbol{\Gamma}} \\ &= \frac{d\boldsymbol{s}}{dt} \cdot \frac{\partial}{\partial \boldsymbol{s}}\left(\frac{m}{2}\left|\boldsymbol{\Omega} \times \boldsymbol{s}\right|^2\right) + mg\boldsymbol{s} \cdot \boldsymbol{\Omega} \times \boldsymbol{\Gamma} \\ &= -\boldsymbol{\Omega} \times \boldsymbol{\lambda} \cdot \dot{\boldsymbol{s}} - \boldsymbol{\kappa} \cdot \boldsymbol{\Omega} \times \boldsymbol{\Gamma}\end{aligned} \tag{3.8}$$

And this is equal to (3.7) so that they cancel out to give $\frac{dE}{dt} = 0$.

# 4 Rolling Ball and Euler's Disk

There are two specialised cases of our system that are two of the widest-studied nonholonomic systems. These are the rolling ball and Euler's disk. The first is a ball rolling with an unbalanced mass distribution and the latter is the spinning and rolling of a flat disk with a radially symmetric mass distribution.

The unbalanced rolling ball may not seem like a specialised version of our system but it is in fact the same system with the additional constraint that the height of the centre of the disk (which is now a ball) is constant (that is, $\boldsymbol{s}_c = r\boldsymbol{\Gamma}$). This is a simple holonomic constraint which we can substitute into our original Lagrangian to give the new Lagrangian:

$$\mathcal{L}_{ball} = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} + \frac{m}{2}\left|\boldsymbol{Y}\right|^2 - mgl\boldsymbol{\chi} \cdot \boldsymbol{\Gamma} - mgr\left|\boldsymbol{\Gamma}\right|^2 \tag{4.1}$$

Since $\left|\boldsymbol{\Gamma}\right|^2 = 1$, the last term is a constant and therefore has no effect on the motion and we may conclude that this Lagrangian is equivalent to the Lagrangian of the rolling ball as defined in [9]. The unbalanced rolling ball is studied extensively in [15]. It is shown that the moment of inertia has a big effect on the motion. If there is a slight asymmetry ($I_1 \neq I_2$) then, as the ball rocks back and forth, it precesses and after a while the direction of the precession changes! *This paper is very readable and I recommend it if you are interested in rolling bodies.*

The simplification to the symmetric disk is as easy as setting $l = 0$. This yields the new Lagrangian:

$$\mathcal{L}_{euler} = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} + \frac{m}{2}\left|\boldsymbol{Y}\right|^2 - mg\boldsymbol{s}_c \cdot \boldsymbol{\Gamma} \tag{4.2}$$

There have been many papers written about the spinning symmetric disk (such as [2, 7, 17, 4]). Many of them look at the physics of the toy Euler's Disk. There is a list of publications on the Euler's Disk website[11] and there is also some information on the Wikipedia page[26]. Many publications focus on how the energy is dissipated and the finite time singularity. Moffatt[18] thought that the reason the disk lost energy and increased frequency was due to viscous properties of the trapped air. Experiments were done spinning disks in a vacuum[22] and also spinning rings which would not trap air. The motion was seen to be generally unaffected leading to the conclusion that Moffatt's theory was inconsistent. The loss of energy was thought to be due to slipping however in the early stages of the motion ($\geq 10°$ from the horizontal) it was shown that the no-slip condition is satisfied so that energy dissipation must be due to rolling friction[19] and the sound and vibration generated. The base on which the toy "Euler's Disk" sits is in fact very smooth and slightly elastic so the vibration is reduced to a minimum and the disk can roll for the optimum amount of time. The spin time is noticeably reduced when the disk is placed on a rougher surface like a tabletop. The base, on which the Euler's disk is set to spin, is also slightly concave. This design was implemented to make the disk want to spin in the center of the base. [13] investigates whether this concavity actually helps increase the spin time of the disk. Spinning of a disk inside a sphere is also further analysed in [14].

# 5 Constraints, Symmetries and Conservation Laws

In 1915 (published in 1918[28]) Emmy Noether, described by Albert Einstein as the most important woman in the history of mathematics[25], proved one of most important theorems in the history of mechanical systems. Noether's theorem describes the link between symmetries of a system and conserved quantities. From Noether's theorem we can link time independent Lagrangians to conservation of energy, translation invariant Lagrangians to conservation of momentum and so on.

## 5.1 Conservation Laws for Chaplygin's Ball and Euler's Disk

As mentioned in section 4, the asymmetric rolling disk can be considered as a generalisation of a rolling ball and a rolling symmetric disk (Euler's disk). We consider the specialised case of an unbalanced rolling ball that is radially symmetric about $\boldsymbol{\chi}$ ,the vector from the center of the ball to the center of mass. This semi-symmetric ball will be referred to as Chaplygin's Ball. Due to the symmetry, its motion is invariant under rotations about the vector $\boldsymbol{\chi}$. We choose the reference coordinates so that $\boldsymbol{E}_3 = \boldsymbol{\chi}$. Both Chaplygin's ball and the symmetric Euler's disk exhibit a symmetry under rotations about the $\boldsymbol{E}_3$ axis. Can we find conservation laws corresponding to this symmetry?

## 5.2 Jellet's Integral

Jellet's integral, $\boldsymbol{\Pi} \cdot \boldsymbol{s}$, is a conserved quantity for Chaplygin's ball. It appears because of the $\boldsymbol{E}_3$ symmetry.

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{\Pi} \cdot \boldsymbol{s}) &= \dot{\boldsymbol{\Pi}} \cdot \boldsymbol{s} + \boldsymbol{\Pi} \cdot \dot{\boldsymbol{s}} \\ &= -\boldsymbol{\Omega} \times \boldsymbol{\Pi} \cdot \boldsymbol{s} + \boldsymbol{\Pi} \cdot \dot{\boldsymbol{s}} \qquad \text{from equations of motion} \end{aligned} \tag{5.1}$$

Now, for Chaplygin's ball, $\boldsymbol{s} = r\boldsymbol{\Gamma} + l\boldsymbol{\chi}$ and so $\dot{\boldsymbol{s}} = -\boldsymbol{\Omega} \times r\boldsymbol{\Gamma}$. Therefore

$$\begin{aligned}\frac{d}{dt}\left(\boldsymbol{\Pi}\cdot\boldsymbol{s}\right) &= \boldsymbol{\Pi}\cdot\boldsymbol{\Omega}\times r\boldsymbol{\Gamma} - \boldsymbol{\Omega}\times\boldsymbol{\Pi}\cdot r\boldsymbol{\Gamma} - \boldsymbol{\Omega}\times\boldsymbol{\Pi}\cdot l\boldsymbol{\chi} \\ &= l\boldsymbol{\chi}\cdot\boldsymbol{\Pi}\times\boldsymbol{\Omega} \\ &= l\left(\Pi_1\Omega_2 - \Pi_2\Omega_1\right) \quad \text{since } \boldsymbol{\chi} = \boldsymbol{E}_3 \\ &= 0\end{aligned}$$

since for Chaplygin's ball, $\mathbb{I} = diag(I_1, I_1, I_3)$ so that $\Pi_1 = I_1\Omega_1$ and $\Pi_2 = I_1\Omega_2$ so that the terms cancel out. What is peculiar is that this quantity is not conserved for Euler's disk, even though both Chaplygin's ball and Euler's disk have the same symmetry. This is due to the non-holonomy of the constraint, which is discussed below. There are conservation laws for Euler's disk but I will not go in to them here since any reference I have found to them has been in terms of Euler angles and this is not in keeping with the layout of this paper. There is an additional constant of motion for Chaplygin's ball known as the Routh constant. I have not found a way of writing it down in anything other than Euler angles so I have omitted it. See [8] for more information.

## 5.3 Nonholonomic Constraints

Nonholonomic constraints generally occur when there is a constraint on the velocities. These constraints are characteristic of any system where rolling is involved. From a mathematical point of view, a nonholonomic constraint is such that the constraint is not integrable. A good physical description of a nonholonomic system is given by [29]: A nonholonomic system is such that the state of the system depends on the path it took to get there. Another interpretation given in [12] is that the system is allowed to move between any two given states without violating the constraint. This is easy to describe with the rolling ball: Given the orientation of a ball, it is possible to roll the ball away from the origin then return it to the origin in any orientation we want, without the ball slipping or sliding. Calculations like "how to roll the ball to get a specific orientation" are known as control theory problems. Nonholonomic control theory problems occur regularly in the field of robot control. Robots on wheels are subject to the nonholonomic rolling constraint and need to be controlled to reach a specific destination in a specific orientation.

Hamilton's principle is not applicable to nonholonomic systems and so, traditionally, nonholonomic systems have been treated with the Lagrange D'Alembert principle as in [4, 32]. This method can be complicated to implement and the alternative Hamilton-Pontryagin approach[3] is more efficient. It also gives us some insight into Noether's theorem for nonholonomic constraints. Recall that as we were taking variations of the action we obtained:

$$\begin{aligned}0 = \delta\mathcal{S} = \int_a^b &\left\langle \frac{\partial\mathcal{L}}{\partial\Omega} - \Pi, \delta\Omega\right\rangle + \left\langle \frac{\partial\mathcal{L}}{\partial\Gamma} - \kappa, \delta\Gamma\right\rangle + \left\langle \frac{\partial\mathcal{L}}{\partial Y} - \lambda, \delta Y\right\rangle \\ &- \left\langle \left(\frac{d}{dt} - \mathrm{ad}^*_\Omega\right)(\Pi - \lambda\diamond s) - \kappa\diamond\Gamma + \lambda\diamond\dot{s}, \eta\right\rangle dt + \Big[\langle \Pi - \lambda\diamond s, \eta\rangle\Big]_a^b \qquad (5.2)\end{aligned}$$

From regular holonomic Noether's theorem we would expect that if some symmetry were to leave the action invariant then we would obtain the conservation law

$$\frac{d}{dt}\langle \Pi - \lambda\diamond s, \eta\rangle = 0 \quad \text{or, in vector form,} \quad \frac{d}{dt}\Big(\big(\boldsymbol{\Pi} - \boldsymbol{\lambda}\times\boldsymbol{s}\big)\cdot\boldsymbol{\eta}\Big) = 0 \qquad (5.3)$$

There must be some kind of link between this quantity and conservation laws. Chaplygin's ball is symmetric about the $\boldsymbol{E}_3$ axis. This $\boldsymbol{E}_3$ symmetry of the Lagrangian leads to conservation of Jellet's integral, $\boldsymbol{\Pi}\cdot\boldsymbol{s}$. We note that this can be expressed in the form

$$\frac{d}{dt}\Big(\big(\boldsymbol{\Pi}-\boldsymbol{\lambda}\times\boldsymbol{s}\big)\cdot\boldsymbol{s}\Big)=0 \tag{5.4}$$

due to anti-symmetry of the vector cross product. But why would this quantity be conserved via symmetry about the $\boldsymbol{E}_3$ axis when it appears more like a symmetry about $\boldsymbol{s}$? The symmetric disk also has this $\boldsymbol{E}_3$ symmetry but Jellet's quantity is not conserved, why is this? These questions are difficult to answer and many people are trying to work their way towards a nonholonomic version of Noether's theorem. Zenkov[32] discusses (linear) conservation laws for nonholonomic systems and shows how it is possible to derive conservation laws for the symmetric Euler's disk. [7] presents a Hamiltonian approach to nonholonomic constraints and defines a kind of alternative form of Noether's theorem, although, rather than relating a symmetry to a conservation law, it describes, in essence, a necessary and sufficient condition for a given quantity to be conserved.

For more information, a relatively light discussion on nonholonomic constraints and how to handle them can be found in [20, 21] or for an extensive study of nonholonomic dynamics with symmetries, with examples, see [1].

## 6 2D Rocking Motion

The simplest solution to the asymmetric disk system is the 2D rocking motion where the disk is just rolling back and forth on its edge oriented in the vertical plane. Obviously, for non-trivial motion we require $l\neq 0$ so the centre of mass is not in the centre of the disk. We show that this system reduces to a second order ODE.

We start by defining the reference configuration of the disk that defines the $(\boldsymbol{E}_1,\boldsymbol{E}_2,\boldsymbol{E}_3)$ vectors. We choose the reference configuration to be the case when the disk is standing on it's side with the $\boldsymbol{E}_2$ axis pointing downwards. With this reference configuration, the spatial vertical vector is given by $\boldsymbol{e}_3=-\boldsymbol{E}_2$.

The rotation of the disk is about the $\boldsymbol{E}_3$ axis so that it can be represented by an element $A(\theta)\in SO(3)$ given by

$$A(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} \tag{6.1}$$

An increase in $\theta$ corresponds to rolling from left to right (see figure 6.1). The left-invariant body angular velocity is calculated to be

$$\Omega=A^{-1}\dot{A}=\begin{pmatrix}0 & -\dot\theta & 0\\ \dot\theta & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\quad\text{with vector form}\quad\boldsymbol{\Omega}=(0,0,\dot\theta)=\dot\theta\boldsymbol{E}_3 \tag{6.2}$$

We usually choose the $(\boldsymbol{E}_1,\boldsymbol{E}_2,\boldsymbol{E}_3)$ coordinates so that the inertia tensor is diagonal. However, in this case we do not need to do this. We make sure that $\boldsymbol{E}_3$ is the principal axis perpendicular to the disk but rotate the $(\boldsymbol{E}_1,\boldsymbol{E}_2)$ axes so that the unit vector to the centre of mass is $\boldsymbol{\chi}=-\boldsymbol{E}_2$. This is an arbitrary choice for simplifying the equations. The vector $\boldsymbol{\chi}$ could point anywhere in the $\boldsymbol{E}_1,\boldsymbol{E}_2$ plane and this simply corresponds to a constant shift in angle $\theta$. We can then

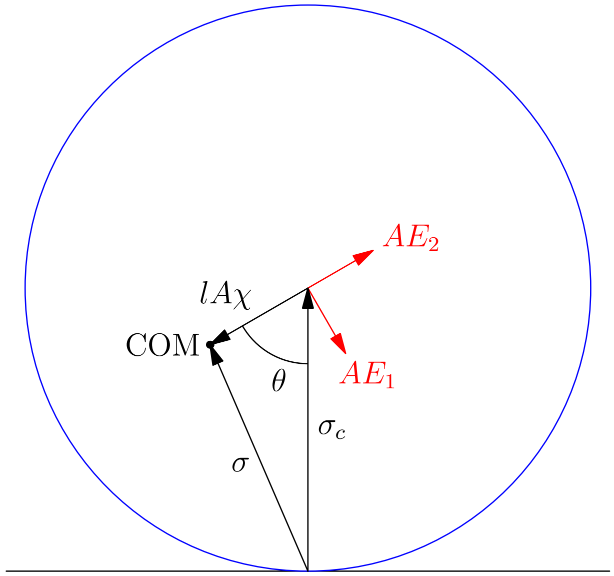



***Figure 6.1:*** *The disk is vertically oriented. $\theta$ is the angle between the vertical and the vector from the center of mass (COM) to the center of symmetry. An increase in $\theta$ corresponds to rolling from left to right.*

easily calculate the other variables/parameters required which are $\mathbf{\Gamma}$, $\boldsymbol{s} = \boldsymbol{s}_c + l\boldsymbol{\chi}$ and the rolling constraint $\boldsymbol{Y} = \Omega \boldsymbol{s}$. Since the disk is circular and remains rolling on it's side, the vector from the point of contact to the centre of symmetry is $\boldsymbol{s}_c = r\mathbf{\Gamma}$ in the body.

$$\mathbf{\Gamma}(\theta) = A^{-1}\boldsymbol{e}_3 = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad \boldsymbol{s} = \begin{pmatrix} r\sin\theta \\ r\cos\theta - l \\ 0 \end{pmatrix}, \quad \boldsymbol{Y} = \mathbf{\Omega} \times \boldsymbol{s} = \dot{\theta} \begin{pmatrix} l - r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \tag{6.3}$$

We now have all the ingredients required to put into the equation of motion:

$$\left(\frac{d}{dt} + \mathbf{\Omega}\times\right)\left(\mathbb{I}\mathbf{\Omega} + m\boldsymbol{s} \times (\mathbf{\Omega} \times \boldsymbol{s})\right) = mg\mathbf{\Gamma} \times \boldsymbol{s} + m\dot{\boldsymbol{s}} \times (\mathbf{\Omega} \times \boldsymbol{s}) \tag{6.4}$$

Calculating components of this equation:

$$\mathbb{I}\mathbf{\Omega} = I_3\dot{\theta}\boldsymbol{E}_3 \implies \mathbf{\Omega} \times \mathbb{I}\mathbf{\Omega} = 0 \tag{6.5}$$

$$\begin{aligned} \boldsymbol{s} \times \boldsymbol{Y} &= \dot{\theta}\left(r^2\sin^2\theta + (r\cos\theta - l)^2\right)\boldsymbol{E}_3 \\ &= \dot{\theta}\left(r^2 + l^2 - 2rl\cos\theta\right)\boldsymbol{E}_3 \\ &= \dot{\theta}\,|\boldsymbol{s}(\theta)|^2\,\boldsymbol{E}_3 \end{aligned} \tag{6.6}$$

$$\begin{aligned} \mathbf{\Gamma} \times \boldsymbol{s} &= (\sin\theta(r\cos\theta - l) - r\sin\theta\cos\theta)\,\boldsymbol{E}_3 \\ &= -l\sin\theta\boldsymbol{E}_3 \end{aligned} \tag{6.7}$$

$$\dot{\boldsymbol{s}} \times \boldsymbol{Y} = \dot{\theta}^2 \begin{pmatrix} r\cos\theta \\ -r\sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} l - r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} = rl\dot{\theta}^2\sin\theta\boldsymbol{E}_3 \tag{6.8}$$

So, combining these, the $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ components of (6.4) are zero. The $\boldsymbol{E}_3$ component gives us an ODE for $\theta$ which can be written as:

$$\frac{d}{dt}\left((I_3 + m\,|\boldsymbol{s}(\theta)|^2)\dot{\theta}\right) = -mgl\sin\theta + mrl\dot{\theta}^2\sin\theta \tag{6.9}$$

which can be rearranged to give

$$\ddot{\theta} = -\frac{ml(g + r\dot{\theta}^2)}{I_3 + m\,|s(\theta)|^2}\sin\theta \tag{6.10}$$

where

$$|s(\theta)|^2 = r^2 + l^2 - 2rl\cos\theta \tag{6.11}$$

This equation is consistent with the equations in [9],[15] where they look at the rocking of Chaplygin's ball in the plane, a rolling ball with $I_1 = I_2$. This is an equivalent problem.

We may compare this ODE to the equation for the simple pendulum which is

$$ml^2\ddot{\theta} = -mlg\sin(\theta) \tag{6.12}$$

where $l$ is the length of the pendulum. We see that the equation for the rolling disk has a more complex expression due to the torque introduced by the rolling constraint[9] and the dependence of the acceleration on the velocity $\dot{\theta}$ not just of position. If $k = r = 0$ for the disk equation then we recover the equation for the simple pendulum but these values are totally non-physical.

Let us compare the motion arising from the rocking disk to that of the simple pendulum. Figure 6.2 shows the phase portraits for both of these systems. We see how for low energies the disk is similar to the pendulum because they are both an approximation of simple harmonic motion (SHM). For larger energies the acceleration of the disk is damped by inertia and so the phase portrait starts to change shape. The figure also shows the homoclinic orbits which occur from the unstable equilibrium when the centre of mass is directly above the centre of symmetry.

We may also look at small amplitude oscillations where $\theta << 1$ so that we ignore higher order terms. In this case the motion simplifies to simple harmonic motion with period $\omega$ defined by:

$$\omega^2 = \frac{mlg}{k + m(r-l)^2} \tag{6.13}$$

When rolling on the horizontal plane, the path of the centre of mass draws out a Curtate Cycloid [23]. This path can be seen in figure 6.3 and has parametric definition

$$x(\theta) = r\theta - l\sin\theta, \quad y(\theta) = r - l\cos\theta \tag{6.14}$$

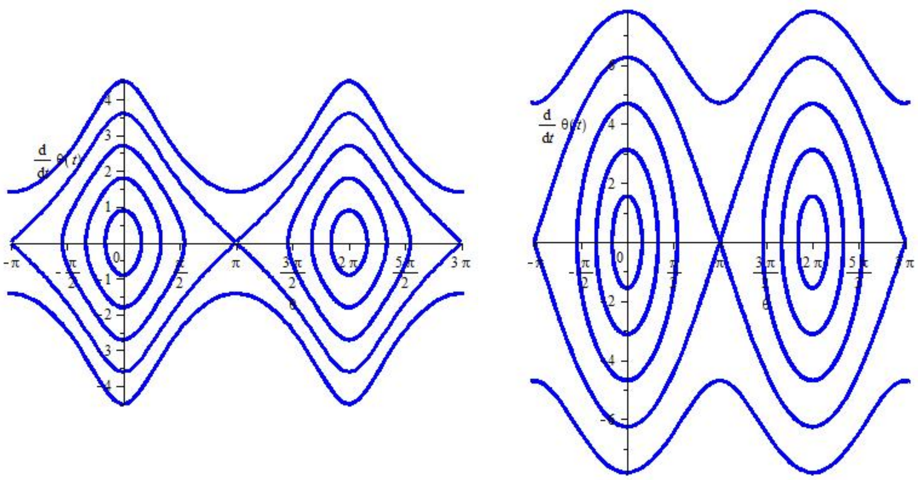
As in the 3D case the energy is still conserved. The energy is given by:

$$E(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) = \frac{1}{2}\langle\boldsymbol{\Omega}, \mathbb{I}\boldsymbol{\Omega}\rangle + \frac{m}{2}|\boldsymbol{\Omega}\times\boldsymbol{s}|^2 + mg\langle\boldsymbol{s}, \boldsymbol{\Gamma}\rangle \tag{6.15}$$

If we substitute in the values of $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ for the 2D motion the conservation of energy equation becomes:

$$E(\theta, \dot{\theta}) = \frac{1}{2}I_3\dot{\theta}^2 + \frac{m}{2}\,|s(\theta)|^2\,\dot{\theta}^2 + mg(r - l\cos\theta) \tag{6.16}$$

We use this below to hint at a Lagrangian structure for this motion.



**(a)** Rolling Disk **(b)** Planar Pendulum

***Figure 6.2:*** *Phase portraits for the rolling vertical disk (a) and the planar pendulum (b). There are similarities for low energies where both systems approximate simple harmonic motion.*

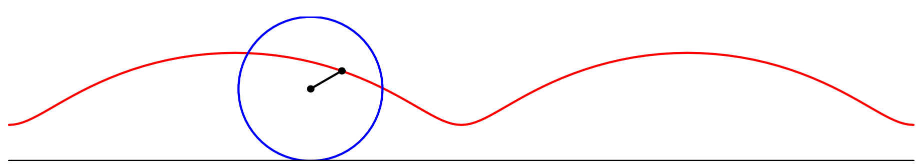


***Figure 6.3:*** *When the vertical disk rolls along a horizontal plane, the center of mass draws out a path known as a* curtate cycloid*.*

## 6.1 Rolling on a Curved Surface

What about rolling on a curved surface? This seemed like an interesting problem to pursue. It is more complicated than it may seem because the point of contact of the disk is not necessarily at the bottom of the disk. The potential energy will now depend on the height of the curve at the point of contact as well as the orientation of the disk. For the disk to roll properly on the curved surface we need to impose the condition that the radius of curvature of the surface is always greater than the radius of the disk so that at any one time, there is only a single point of contact.

To construct the equations of motion for this system we first note that the rolling of the disk on a horizontal plane can be expressed as a one-dimensional system with Lagrangian

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2} I_3 \dot{\theta}^2 + \frac{1}{2} m \left| s(\theta) \right|^2 \dot{\theta}^2 + mgl \cos\theta \tag{6.17}$$

which is simply the sum of the kinetic energies minus the potential energy. The equations of motion are then easily derived from the standard Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \tag{6.18}$$

**Remark** Earlier we mentioned that Hamilton's principle does not apply to nonholonomic systems yet above we have just used it by saying the equations of motion come from the Euler-Lagrange equations. This is not a mistake because the system is holonomic if we restrict to the 2D rolling motion! The easiest way to demonstrate this is by the physical definition outlined in [12] and mentioned earlier in a discussion on nonholonomic constraints. We said that a system is nonholonomic if we can move between any 2 states without violating the constraint. However, for 2D rolling, if we roll the disk away from the origin and back again, it will always have the same orientation. We cannot change the state of the system at the origin without violating the rolling constraint! Hence the system is now holonomic and Hamilton's principle is valid.

□

The move to a curved surface is fairly simple. It is easiest to construct the problem when the curve is defined in parametric coordinates with parameter $\theta$, which will be orientation of the disk relative to the slope (see fig 6.4). We are able to do this because the disk is rolling on the surface. The rolling constraint tells us that the distance travelled along the curve from $\big(x(\theta_0), y(\theta_0)\big)$ to $\big(x(\theta), y(\theta)\big)$ is:

$$r(\theta - \theta_0) = \int_{\theta_0}^{\theta} \sqrt{x'(\phi)^2 + y'(\phi)}\, d\phi \quad \xrightarrow{\text{differentiate w.r.t } \theta} \quad r^2 = x'(\theta)^2 + y'(\theta)^2 \tag{6.19}$$

So we have a curve defined by $\big(x(\theta), y(\theta)\big)$. At any point on the curve we can find the angle of the slope by

$$\alpha(\theta) = \arctan\left(\frac{y'(\theta)}{x'(\theta)}\right) \tag{6.20}$$

As discussed above, the ball must be able to roll on the surface with only a single point of contact. We require that the radius of curvature of the surface is always greater that the radius of the disk. The curvature is defined by

$$\kappa = \left| \frac{1}{r} \frac{d\alpha}{d\theta} \right| \tag{6.21}$$

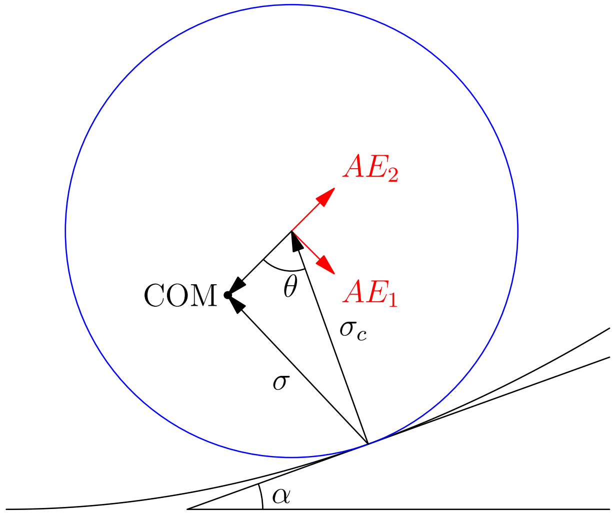
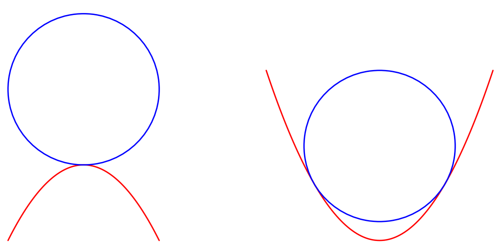



***Figure 6.4:*** *The vertically oriented disk is now rolling on a curved surface. Angle $\theta$ is measured between the normal vector to the curve and the vector from the center of the disk to the center of mass. $\alpha$ is the angle of the slope.*



**(a)** allowed **(b)** disallowed

***Figure 6.5:*** *Here we see the disk rolling on a parabolic curve. The radius of curvature is less than the radius of the disk. However when the curve is concave, as in (a), the disk can still roll unobstructed. When the curve is convex ,as in (b), the ball cannot roll properly since it now has 2 points of contact with the curve.*

The radius of curvature is then $\kappa^{-1}$ so that for the disk to be able to roll on this surface we require

$$\frac{1}{\kappa} > r \implies \left|\frac{d\alpha}{d\theta}\right| < 1 \tag{6.22}$$

This constraint is actually too strict since we only need the surface to have a greater radius of curvature at points where the surface is convex (see figure 6.5). We therefore relax the constraint to be

$$\frac{d\alpha}{d\theta} < 1 \tag{6.23}$$

where an increase in $\theta$ correspond to rolling from left to right.

The Lagrangian for the disk rolling on a slope can now be constructed. The rotational kinetic energy depends on the angular velocity of the disk relative the the fixed vertical axis. This is given by $(\dot{\theta} - \dot{\alpha})$. The rotational kinetic energy can then be written down as

$$KE_{rot} = \frac{1}{2} I_3 \left(\dot{\theta} - \dot{\alpha}\right)^2 = \frac{1}{2} I_3 \left(1 - \alpha'(\theta)\right)^2 \dot{\theta}^2 \tag{6.24}$$

where $I_3$ is the moment of inertia of the disk about the $\boldsymbol{E}_3$ axis perpendicular to the disk. To write down the translational kinetic energy we must first determine the position $\boldsymbol{X} = (X, Y)^T$ of the centre of mass. Using some simple trigonometry we can write down:

$$\boldsymbol{X} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x(\theta) - r\sin\alpha - l\sin(\theta - \alpha) \\ y(\theta) + r\cos\alpha - l\cos(\theta - \alpha) \end{pmatrix} \tag{6.25}$$

$$\implies$$

$$\dot{\boldsymbol{X}} = \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \dot{\theta} \begin{pmatrix} x'(\theta) - r\alpha'(\theta)\cos\alpha - l\left(1 - \alpha'(\theta)\right)\cos(\theta - \alpha) \\ y'(\theta) - r\alpha'(\theta)\sin\alpha + l\left(1 - \alpha'(\theta)\right)\sin(\theta - \alpha) \end{pmatrix} \tag{6.26}$$

The translational kinetic energy is then given by

$$KE_{trans} = \frac{m}{2} \left|\dot{\boldsymbol{X}}\right|^2 \tag{6.27}$$

After lots of paperwork and miraculous cancellations we finally ended up at this nice result:

$$\left|\dot{\boldsymbol{X}}\right|^2 = \left(1 - \alpha'(\theta)\right) |s(\theta)|^2 \tag{6.28}$$

where we have made multiple use of the identities relating $\alpha$ and $x', y'$ as well as the rolling condition $r^2 = x'^2 + y'^2$. As before $|s(\theta)|^2$ is the distance from the point of contact to the centre of mass:

$$|s(\theta)|^2 = r^2 + l^2 - 2rl\cos\theta \tag{6.29}$$

The potential energy is simply $mg$ times the height of the centre of mass. This is given by $Y(\theta)$ in (6.25). We can now write down the Lagrangian:

$$\mathcal{L}(\theta, \dot{\theta}) = \underbrace{\frac{1}{2} I_3 \big(1 - \alpha'(\theta)\big)^2 \dot{\theta}^2}_{\text{rotational KE}} + \underbrace{\frac{m}{2} |s(\theta)|^2 \left(1 - \alpha'(\theta)\right)^2 \dot{\theta}^2}_{\text{translational KE}} - \underbrace{mg\big(y(\theta) + r\cos\alpha - l\cos(\theta - \alpha)\big)}_{\text{potential energy}} \tag{6.30}$$

The Euler-Lagrange equations then lead to the following equation of motion:

$$\frac{d}{dt}\left(\left(I_3+m\,|s(\theta)|^2\right)\left(1-\alpha'(\theta)\right)^2\dot{\theta}\right)$$
$$=\left(1-\alpha'(\theta)\right)\left(mrl\left(1-\alpha'(\theta)\right)\sin\theta-\alpha''(\theta)\left(I_3+m\,|s(\theta)|^2\right)\right)\dot{\theta}^2-V'(\theta) \quad (6.31)$$

where

$$V'(\theta)=mg\Big(y'(\theta)-r\alpha'(\theta)\sin\alpha+l\left(1-\alpha'(\theta)\right)\sin(\theta-\alpha)\Big) \quad (6.32)$$

If $l=0$ these equation simplify to

$$\left(I_3+mr^2\right)\left(1-\alpha'(\theta)\right)\ddot{\theta}=\alpha''(\theta)\left(1-\alpha'(\theta)\right)\left(I_3+mr^2\right)\dot{\theta}^2-mg\left(y'(\theta)-r\alpha'(\theta)\sin\alpha\right) \quad (6.33)$$

As before, the Lagrangian has no explicit time dependence so there is a conserved Energy given by:

$$E(\theta,\dot{\theta})=\frac{1}{2}I_3\left(1-\alpha'(\theta)\right)^2\dot{\theta}^2+\frac{m}{2}\,|s(\theta)|^2\left(1-\alpha'(\theta)\right)^2\dot{\theta}^2+mg\left(y(\theta)+r\cos\alpha-l\cos(\theta-\alpha)\right) \quad (6.34)$$

One question I thought would be interesting to answer is the following: Is there a curved surface such that rolling an unbalanced disk on that surface would give the same motion as a pendulum? Just by looking at the equations of motion we start to think the answer is no (or is it?). For a start there are $\dot{\theta}^2$ terms appearing in the equation of motion which are nowhere to be found in the pendulum. The only way to make these disappear would be to have

$$0=\left(1-\alpha'(\theta)\right)\left(mrl\left(1-\alpha'(\theta)\right)\sin\theta-\alpha''(\theta)\left(I_3+m\,|s(\theta)|^2\right)\right)$$
$$=\frac{\partial}{\partial\theta}\frac{1}{2}\left(\left(I_3+m\,|s(\theta)|^2\right)\left(1-\alpha'(\theta)\right)^2\right) \quad (6.35)$$

In fact this leads to the simplest form of the equations

$$K\ddot{\theta}=-V'(\theta) \quad \text{where} \quad K=\left(I_3+m\,|s(\theta)|^2\right)\left(1-\alpha'(\theta)\right)^2 \quad \text{is a constant} \quad (6.36)$$

We can see that if $l=0$ ($\implies$ $|s(\theta)|$ is constant) then taking $\alpha'(\theta)=$constant will satisfy this relation. Then if $V'(\theta)\sim\sin\theta$ we could get a pendulum. We will see which curve satisfies this later on.

### 6.1.1 Inclined Plane

The simplest example is to consider the symmetric disk $l=0$ rolling on an inclined plane at angle $\alpha$ from the horizontal (see figure 6.6). Then:

$$\alpha(\theta)=\alpha\,(\text{constant}),\quad x'(\theta)=r\cos\alpha,\quad y'(\theta)=r\sin\alpha \quad (6.37)$$

We substitute these into (6.30) to obtain the Lagrangian

$$\mathcal{L}(\theta,\dot{\theta})=\frac{1}{2}I_3\dot{\theta}^2+\frac{m}{2}r^2\dot{\theta}^2-mg\left(r\theta\sin\alpha+r\cos\alpha\right) \quad (6.38)$$

We can now write down the equation of motion from the Euler-Lagrange equations:

$$\ddot{\theta}=-\frac{mgr\sin\alpha}{I_3+mr^2} \quad (6.39)$$

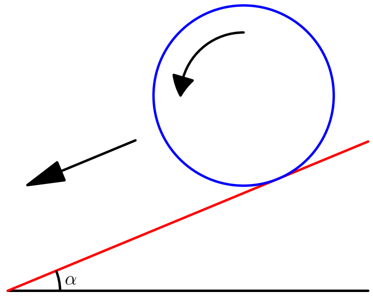



***Figure 6.6:*** *The symmetric disk rolls down an inclined plane at angle $\alpha$ to the horizontal.*

For a uniform disk, $I_3 = \frac{mr^2}{2}$ so we get the equation:

$$\ddot{\theta} = -\frac{2}{3}\frac{g}{r}\sin\alpha \tag{6.40}$$

This is a nice simple equation which shows us that the rotation of the disk is subject to a constant acceleration. Since $\theta$ increases as the disk rolls from left to right (up the slope) it makes sense for the acceleration to be negative - forcing the disk down the slope.

## 6.2 Surface of Stationarity - Constant Potential Energy

Suppose we have an unbalanced disk ($l > 0$). For a given orientation of the disk there must exist a slope such that if placed on this slope the disk will be in equilibrium/remain stationary. So let's set about finding this. Assume that $\dot{\theta} = 0$ then find $y(\theta)$ and $\alpha(\theta)$ such that $\ddot{\theta} = 0$. It is easy to read off from equation (6.31) that we require

$$y'(\theta) - r\alpha'(\theta)\sin\alpha + l\big(1 - \alpha'(\theta)\big)\sin(\theta - \alpha) = 0 \tag{6.41}$$

Note that this is equivalent to solving $V'(\theta) = 0$ so this in fact generates the curve of constant potential energy. This problem on it's own would be difficult to solve but we can use some intuition to help us along the way. For the disk to remain stationary, there must be no overall torque acting on the disk. Therefore we require the centre of mass to be directly above the point of contact. The condition for this to hold is

$$r\sin\alpha + l\sin(\theta - \alpha) = 0 \qquad \Longrightarrow \qquad \alpha(\theta) = \arctan\left(\frac{l\sin\theta}{l\cos\theta - r}\right) \tag{6.42}$$

Plugging this in to (6.41) gives

$$y'(\theta) = -\frac{rl\sin\theta}{|s(\theta)|} \tag{6.43}$$

where

$$|s(\theta)|^2 = r^2 + l^2 - 2rl\cos\theta \tag{6.44}$$

is the length of the vector from the point of contact to the centre of mass. Now using the definition of $\alpha(\theta)$ in (6.20) and (6.42) we can find $x'(\theta)$ to be

$$x'(\theta) = \frac{r(r - l\cos\theta)}{|s(\theta)|} \tag{6.45}$$

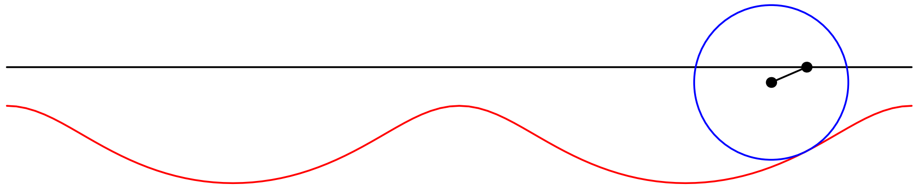


***Figure 6.7:*** *When the ball rolls along this surface, the potential energy remains constant. The black line represents the path of the center of mass. There is no torque acting on the disk since the center of mass is always directly above the point of contact. Since the total energy is conserved there is an exchange of energy between the rotational and translational kinetic energies only.*

It is a simple check to show that these satisfy the rolling condition $r^2 = x'(\theta)^2 + y'(\theta)^2$. These can be integrated to give the curve $\big(x(\theta), y(\theta)\big)$

$$y(\theta) = -|s(\theta)| = -\sqrt{r^2 - 2rl\cos\theta + l^2} \tag{6.46}$$

$$x(\theta) = \int^{\theta} \frac{r^2 - rl\cos\theta}{|s(\theta)|} d\theta \tag{6.47}$$

where the $x(\theta)$ integral is some mess involving elliptic functions. Figure 6.7 shows the surface defined by $\big(x(\theta), y(\theta)\big)$ above. We notice similarities between this and the curtate cycloid generated by the motion of the centre of mass when rolling on a horizontal plane. There is surely some bijective map between the two. We notice that the derivative with respect to $\theta$ of the curve of constant potential energy is the same as the derivative of the curtate cycloid with just a scale factor of $|s(\theta)|$. There is clearly a very close link.

Now lets suppose that we roll the disk along the surface $\big(x(\theta), y(\theta)\big)$ defined above. As mentioned earlier, the potential energy will remain constant. Therefore, if the disk is started rolling, it will remain rolling for all time, it cannot lose kinetic energy.

We need to check that the disk will actually be able to roll freely along the curve with only a single point of contact. We need to check that the radius of curvature satisfies the conditions outlined in (6.23).

$$\frac{d\alpha}{d\theta} = -\frac{l(r\cos\theta - l)}{|s(\theta)|^2} \tag{6.48}$$

So we need to solve:

$$-\frac{\lambda(\cos\theta - \lambda)}{\lambda^2 + 1 - 2\lambda\cos\theta} < 1 \quad \text{for} \quad 0 \leq \lambda = \frac{l}{r} \leq 1 \tag{6.49}$$

Since the denominator is positive we can multiply both sides by it then we get

$$\begin{aligned} &-\lambda(\cos\theta - \lambda) < \lambda^2 + 1 - 2\lambda\cos\theta \\ \Longrightarrow \quad & \lambda\cos\theta < 1 \end{aligned}$$

This is clearly satisfied for all $0 < \lambda < 1$ so this disk will be able to roll on this surface without any problems.

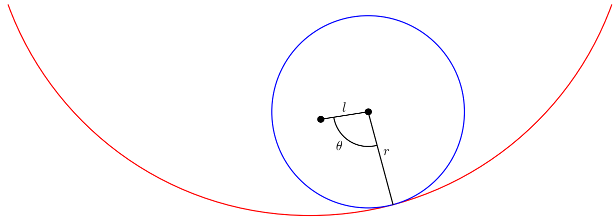



***Figure 6.8:*** *The vertical disk is rolling in the bottom of a large circle of radius R.*

## 6.3 Rolling in the Bottom of a Large Circle

Suppose the surface on which the disk is rolling is the bottom of a large circle of radius $R$ (see figure 6.8). We require $r < R$ for this to be physically possible. Then we can define the curve in parametric co-ordinates with

$$x = R\sin\phi, \quad y = R - R\cos\phi \tag{6.50}$$

Such that $(x, y) = (0, 0)$ at $\phi = 0$. The rolling condition gives us a relation between $\phi$ and $\theta$:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 \quad \Longrightarrow \quad R^2\left(\frac{d\phi}{d\theta}\right)^2 = r^2 \tag{6.51}$$

We therefore take $\phi = \frac{r}{R}(\theta - \theta_0)$ where $\theta_0$ represents the orientation of the disk when it sits at the lowest point in the circular curve (that is $\theta = \theta_0$ when $\phi = 0$). We therefore have

$$x(\theta) = R\sin\left(\frac{r}{R}(\theta - \theta_0)\right), \quad y(\theta) = R - R\cos\left(\frac{r}{R}(\theta - \theta_0)\right) \tag{6.52}$$

so that $\big(x(\theta_0), y(\theta_0)\big) = (0, 0)$. It is now easy to find the angle of the curve as a function of $\theta$:

$$\alpha(\theta) = \arctan\left(\frac{y'(\theta)}{x'(\theta)}\right) = \phi = \frac{r}{R}(\theta - \theta_0) \quad \Longrightarrow \quad \alpha'(\theta) = \frac{r}{R} \tag{6.53}$$

We can now write down the equation of motion:

$$\begin{aligned}\Big(\big(I_3 + m\,|s(\theta)|^2\big)\big(1 - \frac{r}{R}\big)\ddot{\theta}\Big) &= -mrl\left(1 - \frac{r}{R}\right)\dot{\theta}^2\sin\theta \\ &- mg\bigg(r\left(1 - \frac{r}{R}\right)\sin\left(\frac{r}{R}(\theta - \theta_0)\right) + l\left(1 - \frac{r}{R}\right)\sin\left(\theta - \frac{r}{R}(\theta - \theta_0)\right)\bigg)\end{aligned} \tag{6.54}$$

Noticing that if we set $l = 0$ then $|s(\theta)|^2 = r^2$ so that the motion equation becomes

$$\left(I_3 + mr^2\right)\left(1 - \frac{r}{R}\right)^2\ddot{\theta} = -mgr\left(1 - \frac{r}{R}\right)\sin\left(\frac{r}{R}(\theta - \theta_0)\right) \tag{6.55}$$

We recognise this to be the equation for the planar pendulum! So the symmetric disk rolling in a large cylinder has the same motion as a pendulum, irrespective of the radius of the cylinder! Such a beautiful result!!! To look at this in more detail we write the equation in terms of $\phi = \frac{r}{R}(\theta - \theta_0)$ so that $\ddot{\theta} = \frac{R}{r}\ddot{\phi}$

$$\left(I_3 + mr^2\right)\left(1 - \frac{r}{R}\right)\frac{R}{r}\ddot{\phi} = -mgr\sin\phi \tag{6.56}$$

A solid disk of mass $m$, radius $r$ with radially symmetric density has moment of inertia $I_3 = kmr^2$[27] where $0 \leq k \leq 1$. $k = 0$ represents a point mass and $k = 1$ represents a ring of radius $r$. The motion equations now become

$$\ddot{\phi} = -\frac{g}{(k+1)(R-r)} \sin\phi \tag{6.57}$$

Here we now realise that the centre of mass is a distance $(R - r)$ from the centre of the circle and its position is determined by angle $\phi$. Therefore the centre of the disk moves just like a pendulum in a gravitational field of strength $\frac{g}{k+1}$. When $k = 0$, a point mass with no inertia, we return to the regular pendulum. The most common case would be for a uniform disk ($k = 0.5$). Then the motion of the centre of mass is equivalent to the motion of a pendulum of length $R-r$ in a gravitational field of strength $g_c = \frac{2}{3}g$.

Alternatively, for a physical comparison, consider a pendulum of length $L = (k+1)(R-r)$. If we put this next to the symmetric disk rolling inside a circle their motion should be equivalent in terms of period and angle.

**Remark** If a disk in a circle is a planar pendulum then surely a ball rolling inside a large, hollow sphere is the spherical pendulum. Proving this would be an interesting direction for some further work.

□

The following question arose during discussions with my supervisor: If the symmetric disk rolling in a circle is a pendulum, would a symmetric disk rolling in a parabola be a harmonic oscillator? It seemed to be entirely possible.

Imagine a particle, under the influence of gravity, sliding without friction in a parabola. Then the potential energy is $V = mgz = mgx^2$ where $z$ is the height of the particle. Then Newton's equations of motion give

$$\ddot{x} = -\frac{1}{2}mgx$$

It is a motion equation of the type $\ddot{x} = -kx$ which is the definition of Simple Harmonic Motion (SHM).

**Remark** It is the $x$ position of the point of contact which we would want to follow SHM. I tried to put the parabolic curve into the equations of motion for the disk and transformed to the equation of motion for the $x$ position of the point of contact. However, I did not arrive at anything close to simple harmonic motion. I then realised the reason that this breaks down: The center of mass does not follow the path of a parabola! While the point of contact does follow a parabola, the center of mass follows a path parallel to a parabola, which is therefore not a parabola! When the disk rolls in a circle, both the center of mass and the the point of contact follow a circular path.

It must be noted that there could still exist a curve such that when rolled in this curve we get simple harmonic motion for $x$ position of the center of mass. A worthwhile try would be the curve such that the center of mass does follow the path of a parabola. The equation of the curve a distance $r$ from a parabola $y = \frac{1}{2}kx^2$ is given by:

$$\tilde{x} = x \pm \frac{rkx}{\sqrt{1+k^2x^2}}, \quad \tilde{y} = \frac{1}{2}kx^2 \mp \frac{r}{\sqrt{1+k^2x^2}}$$

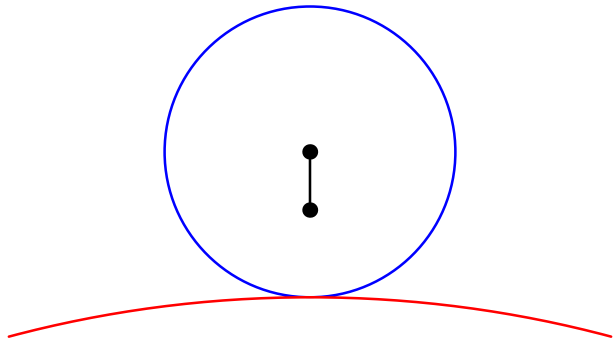


***Figure 6.9:*** *If an asymmetric disk sits vertically with its center of mass downwards on top of circular curve, it will sometimes be unstable and sometimes be stable, depending on the radius of the curve on which it sits.*

As found on Wolfram Mathworld [24]. So if the centre of the disk follows a parabola $y = \frac{1}{2}kx^2$, the point of contact will have position $(\tilde{x}, \tilde{y})$ defined by the equations above. Further work must be done to see if this curve will generate SHM.

□

## 6.4 Stability on Top of a Circle

Suppose we place an unbalanced ($l > 0$) disk on top of a cylinder of radius $R$ so that the centre of mass is at the bottom (see figure 6.9). Clearly, if the disk is placed on a flat surface ($R = \infty$), it would be in a stable equilibrium. However if the disk were on top of a point ($R = 0$) it would be in unstable equilibrium. Therefore there must exist a critical radius of the cylinder $R_c$ such that the equilibrium switches from stable ($R > R_c$) to unstable ($R < R_c$). Let's find it!

The curve is defined by

$$x(\theta) = R\sin\left(\frac{r}{R}\theta\right), \quad y(\theta) = R\cos\left(\frac{r}{R}\theta\right) - R, \quad \alpha(\theta) = \arctan\left(\frac{y'(\theta)}{x'(\theta)}\right) = -\frac{r}{R}\theta \tag{6.58}$$

The key to working out the stability is to look at the potential energy. This is

$$\begin{aligned} V(\theta) &= y(\theta) + r\cos\alpha - l\cos(\theta - \alpha) \\ &= R\cos\left(\frac{r}{R}\theta\right) - R + r\cos\left(-\frac{r}{R}\theta\right) - l\cos\left(\left(1 + \frac{r}{R}\right)\theta\right) \end{aligned} \tag{6.59}$$

There is clearly a maximum or minimum at $\theta = 0$ (simple check that $V'(0) = 0$). To find out whether this is a maximum (unstable) or a minimum (stable) we must look at the second derivative.

$$V'(\theta) = -r\left(1 + \frac{r}{R}\right)\sin\left(\frac{r}{R}\theta\right) + l\left(1 + \frac{r}{R}\right)\sin\left(\left(1 + \frac{r}{R}\right)\theta\right) \tag{6.60}$$

$$V''(\theta) = -\frac{r^2}{R^2}(R + r)\cos\left(\frac{r}{R}\theta\right) + \frac{l}{R^2}(R + r)^2\cos\left(\left(1 + \frac{r}{R}\right)\theta\right) \tag{6.61}$$

The equilibrium is therefore unstable if $V''(0) < 0$ (maximum) and stable if $V''(0) > 0$ (minimum). We therefore define the critical radius $R_c$ to be when $V''(0) = 0$. This is as simple as

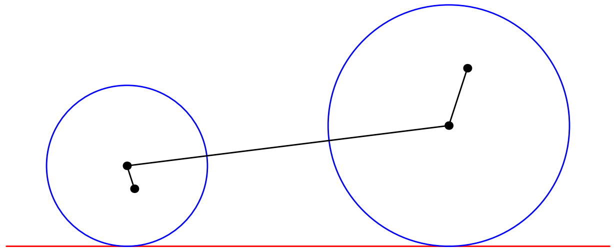


***Figure 6.10:*** *Two vertical disks on a horizontal plane are joined by a light inextensible rod connected to the centers of the disks.*

solving a linear equation. We get:

$$-\frac{r^2}{R_c^2}(R_c + r) + \frac{l}{R_c^2}(R_c + r)^2 = 0 \quad \Longrightarrow \quad R_c = \frac{r(r-l)}{l} \tag{6.62}$$

## 6.5 Two Coupled Rollers

Suppose we have two disks rolling on a horizontal plane with Lagrangians

$$\mathcal{L}_i(\theta_i, \dot{\theta}_i) = \frac{1}{2}k_i\dot{\theta}_i^{\,2} + \frac{1}{2}m_i\,|s_i(\theta_i)|^2\,\dot{\theta}_i^{\,2} + m_i g l_i \cos\theta_i, \quad i = 1, 2 \tag{6.63}$$

We impose the constraint that the centres of the disks are joined together by a light inextensible rod of length $L$, somewhat like a bicycle or car would have two axles rigidly joined together (see figure 6.10). Since they are rolling on a flat surface this constraint is equivalent to saying that the distance rolled by each of the disks is the same. The length of the rod will have no effect on the motion. The constraint is written as:

$$r_1\big(\theta_1 - \theta_1(0)\big) = r_2\big(\theta_2 - \theta_2(0)\big)$$

We could impose this constraint using a Lagrange multiplier:

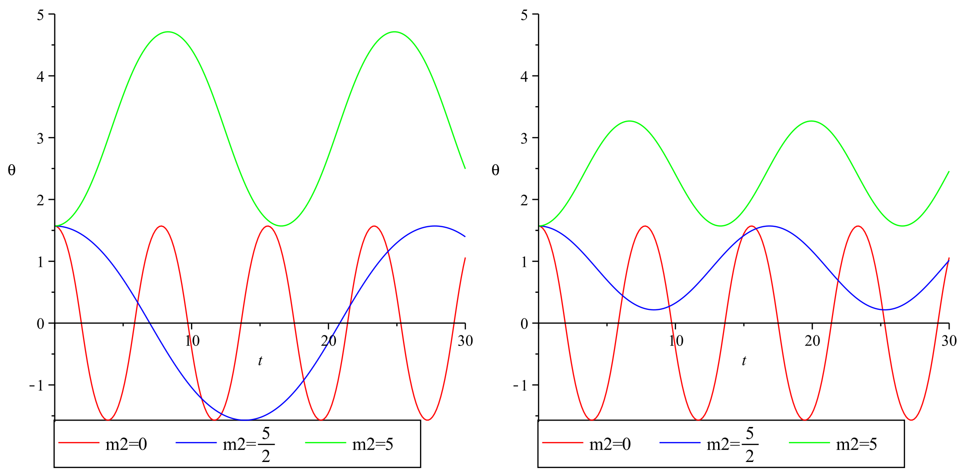
$$\mathcal{L}(\theta_1, \dot{\theta}_1, \theta_2, \theta_2) = \mathcal{L}_1(\theta_1, \dot{\theta}_1) + \mathcal{L}_2(\theta_2, \dot{\theta}_2) + \mu\Big(r_1\big(\theta_1 - \theta_1(0)\big) - r_2\big(\theta_2 - \theta_2(0)\big)\Big) \tag{6.64}$$

However, the constraint is simple and it is much easier to use the following one-dimensional reduced Lagrangian for the first disk and then reconstruct the solution for the second disk:

$$\mathcal{L}(\theta_1, \dot{\theta}_1) = \mathcal{L}_1(\theta_1, \dot{\theta}_1) + \mathcal{L}_2\left(\frac{r_1}{r_2}\big(\theta_1 - \theta_1(0)\big) + \theta_2(0), \frac{r_1}{r_2}\dot{\theta}_1\right) \tag{6.65}$$

where we have simply used the constraint to write $\theta_2$ in terms of $\theta_1$ and substituted this into $\mathcal{L}_2$. This works because the constraint is simple and linear. Once we solve this to find $\theta_1(t)$ we simply reconstruct $\theta_2(t)$ using

$$\theta_2(t) = \frac{r_1}{r_2}\big(\theta_1(t) - \theta_1(0)\big) + \theta_2(0)$$




**(a)** 180° out of phase **(b)** 198° out of phase

***Figure 6.11:*** *An asymmetric disk of mass $m_1 = 3$ is rolling in the plane in a periodic motion. Another disk of the same radius, same mass distribution but different total mass $m_2$ is joined to its center via a light inextensible rod. They both roll on the plane in a periodic motion. The graphs above show the orientation of the first disk with various values of $m_2$. In (a) the disks are totally out of phase and as a consequence, the amplitude of oscillation remains the same with the addition of the second disk. When $m_2 < m_1$ the period of the oscillation increases as $m_2$ increases. If $m_2 > m_1$ then the second disk becomes the dominant driving force and the first disk rolls in the other direction. The period of the oscillation now decreases with an increases of $m_2$. In (b) the disks are slightly out of phase and, as a consequence, the amplitude of oscillation changes.*

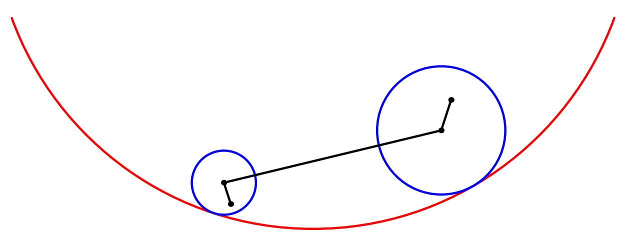


***Figure 6.12:*** *Two vertical disks rolling in the bottom of a circle are joined by a light inextensible rod connected to the centers of the disks.*

## 6.6 Two Disks Rolling in the Bottom of a Large Circle

Imagine now that our 2 connected disks are rolling in the bottom of a large circle (see figure 6.12). The Lagrangian for each disk is given by

$$\mathcal{L}_i(\theta_i, \dot{\theta}_i) = \frac{1}{2}k_i\dot{\theta}_i^2 + \frac{1}{2}m_i \left|s_i(\theta_i)\right|^2 \dot{\theta}_i^2 - m_i g\big(y(\theta_i) + r_i \cos\alpha_i - l_i \cos(\theta_i + \alpha_i)\big) \tag{6.66}$$

The curve on which they are rolling is defined by

$$x(\theta_i) = R\sin\left(\frac{r_i}{R}(\theta_i - \theta_i(0))\right) \tag{6.67}$$

$$y(\theta_i) = R - R\cos\left(\frac{r_i}{R}(\theta_i - \theta_i(0))\right) \tag{6.68}$$

$$\alpha(\theta_i) = \arctan\left(\frac{y'(\theta_i)}{x'(\theta_i)}\right) = \frac{r_i}{R}(\theta_i - \theta_i(0)) \tag{6.69}$$

The constraint we want to impose is that the distance between the centres of the disks is constant. The position of the centre of disk $i$ is

$$\boldsymbol{X}_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} (R - r_i)\sin\alpha_i \\ R - (R - r_i)\cos\alpha_i \end{pmatrix} \tag{6.70}$$

We impose the constraint that $\left|\boldsymbol{X}_1 - \boldsymbol{X}_2\right|^2 = \Upsilon^2$ is constant. Expanding this norm we get

$$(R - r_1)^2 + (R - r_2)^2 - 2(R - r_1)(R - r_2)\cos(\alpha_1 - \alpha_2) = \Upsilon^2 \tag{6.71}$$

So for this to be satisfied we require $\alpha_1 - \alpha_2 = \omega$ to be a constant. If we are given the distance between the centres, $\Upsilon$, then

$$\omega = \alpha_1 - \alpha_2 = \arccos\left[\frac{(R - r_1)^2 + (R - r_2)^2 - \Upsilon^2}{2(R - r_1)(R - r_2)}\right] \tag{6.72}$$

We notice that this is just a simple linear constraint between $\theta_1$ and $\theta_2$. We can therefore use the same method as for rolling on a flat surface. We write $\theta_2(t)$ in terms of $\theta_1(t)$:

$$\theta_2\big(\theta_1(t)\big) = \frac{r_1}{r_2}\big(\theta_1(t) - \theta_1(0)\big) - \frac{\omega R}{r_2} + \theta_2(0) \tag{6.73}$$

We then write down the reduced Lagrangian for $\theta_1(t)$:

$$\mathcal{L}(\theta_1, \dot{\theta}_1) = \mathcal{L}_1(\theta_1, \dot{\theta}_1) + \mathcal{L}_2\left(\theta_2\big(\theta_1(t)\big), \frac{d}{dt}\theta_2\big(\theta_1(t)\big)\right) \tag{6.74}$$

We can then solve this one-dimensional system from the Euler-Lagrange equations to obtain $\theta_1(t)$ and use (6.73) to reconstruct the equation for $\theta_2(t)$.

Figure 6.13 shows an example of a large symmetric disk with a small asymmetric disk attached to it. We see how the motion of the first disk, which would normally be a pendulum on it's own, is affected by the increase of mass of the second smaller disk. The amplitude of the oscillations is reduced while the over all period is increased. The fact that the second disk is smaller and is asymmetric means that the overall oscillations become wavy, which corresponds to the phase of the second disk.

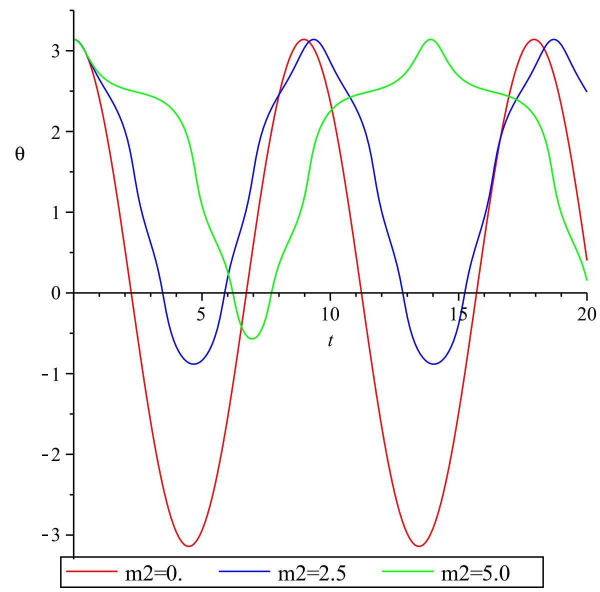



***Figure 6.13:*** *A symmetric disk of radius* $r_1 = 2$ *and mass* $m_1 = 2$ *rolls in the bottom of a circle of radius* $R = 10$ *under the influence of gravity. A second, asymmetric disk of mass* $m_2$ *and radius* $r_2 = 0.5$ *is attached to the first disk via a rod of length* $3$ *joining their centres. The graph shows how the motion of the first disk is affected by the addition of the second disk for different masses. When* $m_2 = 0$ *the disk rolls like a pendulum.*

**Remark** The problem of coupled disks rolling on a general curved surface is a very difficult one. There would no longer be a simple linear relation between the orientations of the disks. Even writing down the equations of motion is probably a very difficult task. The place to start would be to implement the constraint via a Lagrange multiplier:

$$\mathcal{L}(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \mathcal{L}_1(\theta_1, \dot{\theta}_1) + \mathcal{L}_2(\theta_2, \dot{\theta}_2) + \mu \left( |\boldsymbol{X_1}(\theta_1) - \boldsymbol{X_2}(\theta_2)|^2 - \Upsilon^2 \right)$$

□

# 7 Regular Motion

When we spin Euler's disk we don't actually see most of the possible motions which are allowed by the equations of motion. This is because, in our non-ideal world, there is slippage (see [17]). If we observe the real-life motion of the symmetric disk we see that the point of contact seems to draw out a circle and the angle of the disk to the horizontal appears constant (until the disk loses energy due to vibrations and slippage). Therefore it makes sense to analyse this particular situation. In this case the potential energy stays constant. This analysis is done in papers such as [17]. However, that paper uses Euler angles so can simply set $\theta(t) = \theta_0$. To implement this constraint on the Geometric form of the equations we use Lagrange Multipliers. The new constrained Lagrangian becomes

$$S = \int \mathcal{L}(\boldsymbol{\Omega}, \boldsymbol{Y}, \boldsymbol{\Gamma}) + \mu \left( \boldsymbol{s} \cdot \boldsymbol{\Gamma} - \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0 \right) dt \tag{7.1}$$

where $\mu$ is the Lagrange multiplier. The new equations of motion become:

$$\left( \frac{d}{dt} + \boldsymbol{\Omega} \times \right) \left( \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{s} \times (\boldsymbol{\Omega} \times \boldsymbol{s}) \right) = (\mu - mg)\boldsymbol{s} \times \boldsymbol{\Gamma} + m\boldsymbol{s} \times (\boldsymbol{\Omega} \times \boldsymbol{s}) \tag{7.2}$$

After a bit or rearrangement we get:

$$(\mu - mg)\boldsymbol{s} \times \boldsymbol{\Gamma} = \left( \frac{d}{dt} + \boldsymbol{\Omega} \times \right) \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{s} \times (\dot{\boldsymbol{\Omega}} \times \boldsymbol{s} + \boldsymbol{\Omega} \times \dot{\boldsymbol{s}}) + m\boldsymbol{\Omega} \times \left( \boldsymbol{s} \times (\boldsymbol{\Omega} \times \boldsymbol{s}) \right) \tag{7.3}$$

The next challenge is to find the value of the Lagrange multiplier. We use the spherical pendulum problem, with the constraint of constant length, as an analogous situation. To find the Lagrange multiplier of the spherical pendulum we use the constraint *and* it's first and second derivatives. To find the value of the Lagrange multiplier for the rolling disk we take the inner product of the above equation with $\boldsymbol{s} \times \boldsymbol{\Gamma}$ to make it into a scalar equation and then substitute in the constraint and it's first 2 derivatives. These are:

$$\boldsymbol{s}(t) \cdot \boldsymbol{\Gamma}(t) = \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0 \tag{7.4}$$

$$\boldsymbol{s} \cdot \dot{\boldsymbol{\Gamma}} = \boldsymbol{s} \cdot \boldsymbol{\Gamma} \times \boldsymbol{\Omega} = 0 \tag{7.5}$$

$$\dot{\boldsymbol{s}} \cdot \boldsymbol{\Gamma} \times \boldsymbol{\Omega} + \boldsymbol{s} \cdot \dot{\boldsymbol{\Gamma}} \times \boldsymbol{\Omega} + \boldsymbol{s} \cdot \boldsymbol{\Gamma} \times \dot{\boldsymbol{\Omega}} = 0 \tag{7.6}$$

where, in the second equation, we use the fact that $\dot{\boldsymbol{s}} \cdot \boldsymbol{\Gamma} = 0$. We now need to take the inner product of each element of (7.3) with $\boldsymbol{s} \times \boldsymbol{\Gamma}$:

$$\boldsymbol{\Omega} \times \left( \boldsymbol{s} \times (\boldsymbol{\Omega} \times \boldsymbol{s}) \right) \cdot (\boldsymbol{s} \times \boldsymbol{\Gamma}) = \boldsymbol{s}^2 (\boldsymbol{s} \cdot \boldsymbol{\Omega}) \boldsymbol{\Omega} \cdot \boldsymbol{\Gamma} - (\boldsymbol{s} \cdot \boldsymbol{\Omega})^2 \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0$$

$$\boldsymbol{s} \times (\boldsymbol{\Omega} \times \dot{\boldsymbol{s}}) \cdot (\boldsymbol{s} \times \boldsymbol{\Gamma}) = (\boldsymbol{s} \cdot \dot{\boldsymbol{s}}) \boldsymbol{\Omega} \cdot \boldsymbol{s} \times \boldsymbol{\Gamma} - (\boldsymbol{s} \cdot \boldsymbol{\Omega}) \dot{\boldsymbol{s}} \cdot \boldsymbol{s} \times \boldsymbol{\Gamma} = -(\boldsymbol{s} \cdot \boldsymbol{\Omega}) \dot{\boldsymbol{s}} \cdot \boldsymbol{s} \times \boldsymbol{\Gamma}$$

$$\boldsymbol{s} \times (\dot{\boldsymbol{\Omega}} \times \boldsymbol{s}) \cdot (\boldsymbol{s} \times \boldsymbol{\Gamma}) = \boldsymbol{s}^2 \dot{\boldsymbol{\Omega}} \cdot \boldsymbol{s} \times \boldsymbol{\Gamma} = \boldsymbol{s}^2 \left( \boldsymbol{\Omega} \times \dot{\boldsymbol{\Gamma}} \cdot \boldsymbol{s} + \boldsymbol{\Omega} \times \boldsymbol{\Gamma} \cdot \dot{\boldsymbol{s}} \right)$$

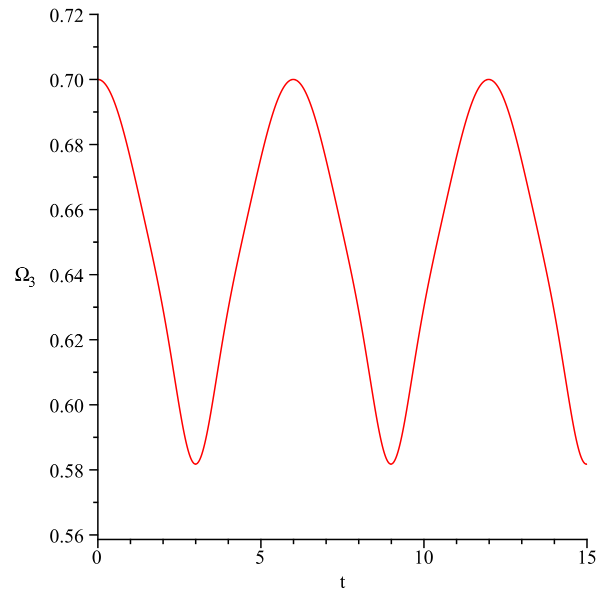



***Figure 7.1:*** *Plot showing the oscillation of the angular velocity about the axis perpendicular to the disk (unbalanced disk only). This is one of the key visual differences noticed when comparing the unbalanced disk to the symmetric disk.*

The first equation uses the constraint, the second equation uses the first derivative of the constraint and the last equation uses the second derivative of the constraint. We can now combine this to find an equation for the Lagrange multiplier:

$$\begin{aligned}\left(\boldsymbol{s}^2 - (\boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0)^2\right)(\mu - mg) = (\boldsymbol{s} \times \boldsymbol{\Gamma}) \cdot \left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right) \mathbb{I}\boldsymbol{\Omega} + m(\boldsymbol{s} \cdot \boldsymbol{\Omega})\boldsymbol{s}^2\boldsymbol{\Omega} \cdot \boldsymbol{\Gamma} - (\boldsymbol{s} \cdot \boldsymbol{\Omega})^2 \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0 \\ - m(\boldsymbol{s} \cdot \boldsymbol{\Omega})\dot{\boldsymbol{s}} \cdot \boldsymbol{s} \times \boldsymbol{\Gamma} + m\boldsymbol{s}^2 \left(\boldsymbol{\Omega} \times \dot{\boldsymbol{\Gamma}} \cdot \boldsymbol{s} + \boldsymbol{\Omega} \times \boldsymbol{\Gamma} \cdot \dot{\boldsymbol{s}}\right) \quad (7.7)\end{aligned}$$

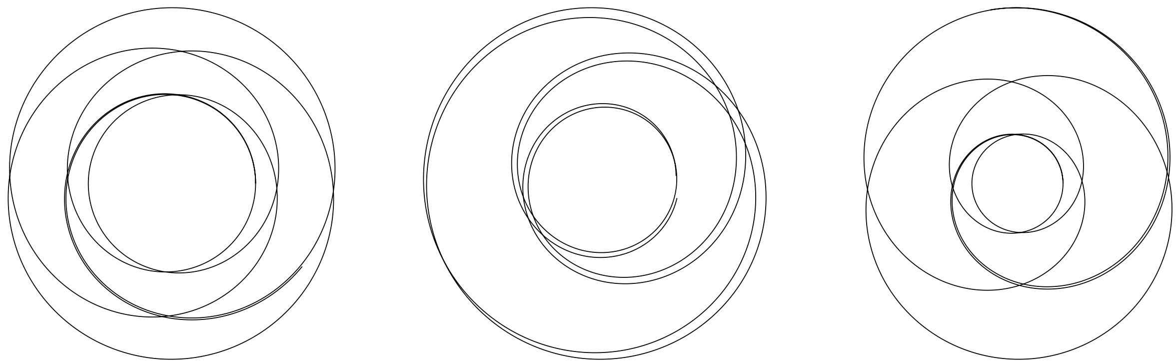
Not a very nice expression but it embeds all the information about the constraint and its derivatives.

We integrate the equations numerically (as discussed in section 10) and we see that for the symmetric disk the path of the point of contact becomes a circle, consistent with [17]. The path of the point of contact for the asymmetric disk is very different and some examples can be seen in figure 7.2.

An interesting feature of the asymmetric disk under regular motion is that the $\boldsymbol{E}_3$ component of the body angular velocity oscillates (see figure 7.1). This is one of the first visual differences we noticed when spinning the asymmetric Euler disk and comparing it to the symmetric Euler disk. We have managed to show its presence numerically from the equations of motion. Success! In the symmetric case this angular velocity stays constant under regular motion.

## 7.1 Constraint via Penalty Functions

It is also possible to use a method called penalty functions to apply a more relaxed version of the constraint. With Lagrange multipliers the motion is restricted to the constraint distribution.



**(a)** **(b)** **(c)**

***Figure 7.2:*** *The path of the point of contact for regular motion of the asymmetric disk with varying initial conditions. There is a clear divergence from the circular path drawn out by the symmetric disk. The paths do exhibit some symmetries which suggests that periodic motions could occur for certain initial conditions.*

With penalty functions the motion is allowed to deviate from the constraint distribution but is penalised for doing so in a way that tries to bring the motion back to the constraint distribution. We put the square of the constraint (always positive) into the Lagrangian and instead of using a Lagrange multiplier we multiply it by a large constant. Then when we try to minimise/maximise the action functional in Hamilton's principle, the large coefficient to the constraint means that the motion must stay close to the constraint distribution. So the constraint is almost satisfied.

Here is a simple example: Suppose we have a Lagrangian $L\big(q(t), \dot{q}(t)\big)$ and we want to implement the constraint $f(q) = 0$ by penalty functions, then we look at stationarity of the action functional:

$$S = \int_a^b \mathcal{L}_c\big(q(t), \dot{q}(t)\big) dt = \int_a^b \mathcal{L}\big(q(t), \dot{q}(t)\big) \pm \frac{1}{2\epsilon^2} \left|f(q)\right|^2 dt \tag{7.8}$$

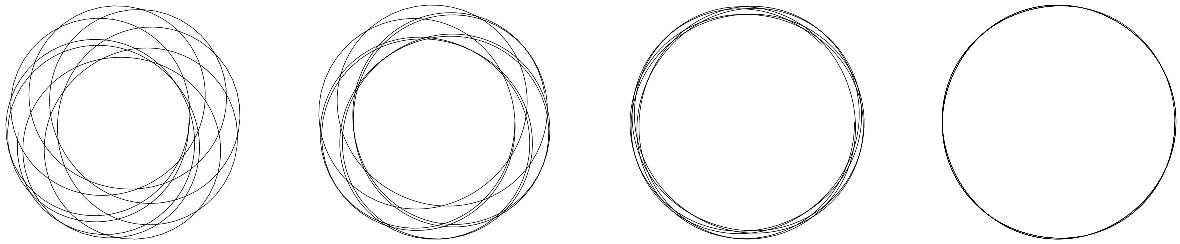
where $\epsilon \in \mathbb{R}$ is small. Notice that we use the norm $|\cdot|^2$. This is because $f(q)$, in general, may not be a scalar valued function so that $|\cdot|^2$ becomes a positive definite non-degenerate metric. The choice of sign depends on whether the action is being maximised or minimised.

In the case of the disk, our constraint is $\boldsymbol{s} \cdot \boldsymbol{\Gamma} - \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0 = 0$. If we put this into the action functional using penalty functions we get

$$S = \int \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} + \frac{m}{2} \left|\boldsymbol{Y}\right|^2 - mg\boldsymbol{s} \cdot \boldsymbol{\Gamma} - \frac{m}{2\epsilon^2} \left(\boldsymbol{s} \cdot \boldsymbol{\Gamma} - \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0\right)^2 dt \tag{7.9}$$

We now take variations like we did before to obtain the equations of motion. The only term which changes from the original equations is $\boldsymbol{\kappa} = \partial\mathcal{L}_c/\partial\boldsymbol{\Gamma}$. It becomes

$$\begin{aligned} \boldsymbol{\kappa} = \frac{\partial \mathcal{L}_c}{\partial \boldsymbol{\Gamma}} &= -mg\boldsymbol{s} - \frac{m}{\epsilon^2} \left(\boldsymbol{s} \cdot \boldsymbol{\Gamma} - \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0\right) \boldsymbol{s} \\ &= -m \left(g + \frac{1}{\epsilon^2} \left(\boldsymbol{s} \cdot \boldsymbol{\Gamma} - \boldsymbol{s}_0 \cdot \boldsymbol{\Gamma}_0\right)\right) \boldsymbol{s} \end{aligned} \tag{7.10}$$



**(a)** No Penalty **(b)** $\epsilon = 0.5$ **(c)** $\epsilon = 0.2$ **(d)** $\epsilon = 0.1$

***Figure 7.3:*** *The path of the point of contact of a symmetric disk with different levels of penalty. As we increase the penalty (decrease $\epsilon$) the path tends towards a circle, characteristic of regular motion.*

So the penalty acts like a rescaling of the gravitational field when the height of the centre of mass deviates from the initial height. If the COM goes higher than the starting height then the effective gravitational field increases to pull it back down. If the COM goes lower, the effective gravity decreases.

The new equations of motion are:

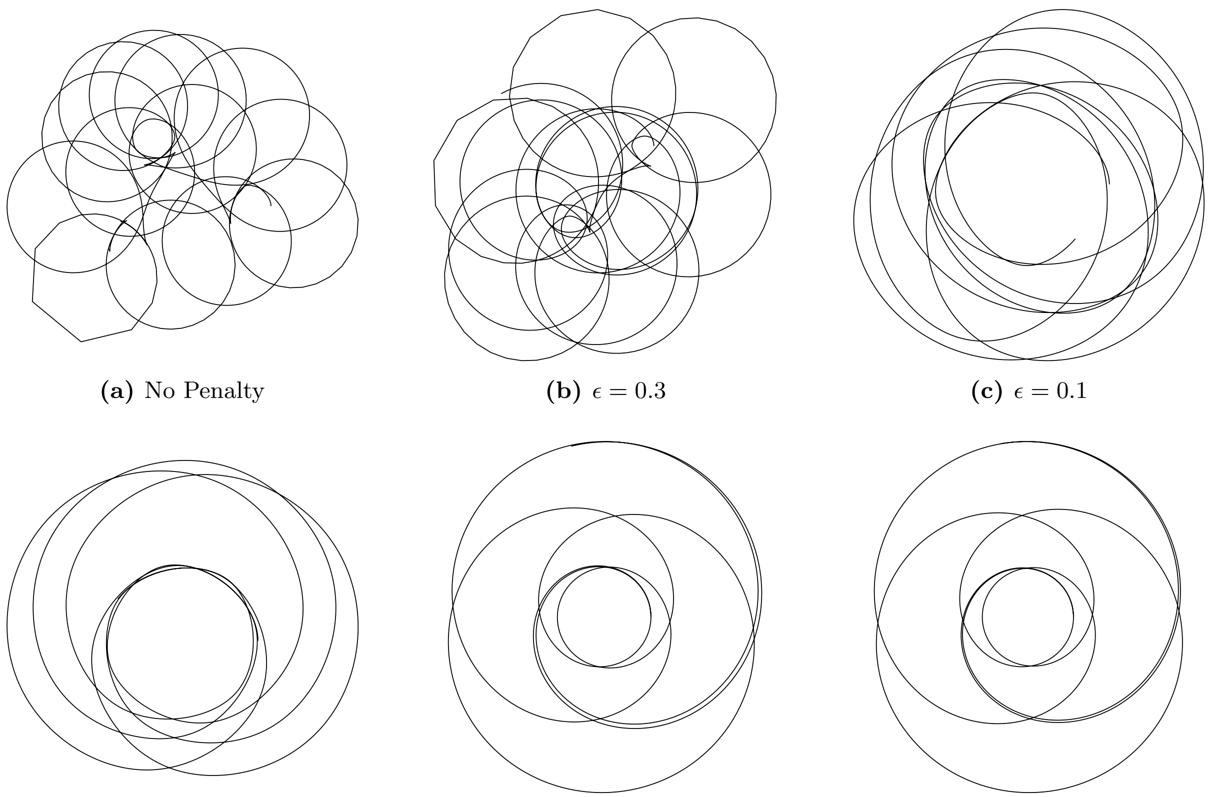
$$\left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\left(\mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{s}\times(\boldsymbol{\Omega}\times\boldsymbol{s})\right) = -m\left(g + \frac{1}{\epsilon^2}\left(\boldsymbol{s}\cdot\boldsymbol{\Gamma} - \boldsymbol{s}_0\cdot\boldsymbol{\Gamma}_0\right)\right)\boldsymbol{s}\times\boldsymbol{\Gamma} + m\dot{\boldsymbol{s}}\times(\boldsymbol{\Omega}\times\boldsymbol{s}) \quad (7.11)$$

Figures 7.3 and 7.4 demonstrate the effect of the penalty on the path of the point of contact for the symmetric and asymmetric disk respectively. The motion of the symmetric disk is fairly conservative anyway so the penalty does not need to be that high to bring the disk back to a circular, regular motion. The motion of the unconstrained asymmetric disk is extremely erratic as can be seen in figure 7.4a. This motion is so chaotic that we even start to see the effect of the discretisation in the plots where the disk is moving very fast and the curve is no longer smooth. The asymmetric disk requires a very strict penalty to bring the motion back to something similar to regular motion.

# 8 Approach to Flat

When we observe the symmetric Euler's disk we see that the rate at which the point of contact rotates, or wobble frequency, increases as the disk gets closer to the flat. In this section I will try and reach some kind of description of the rate at which this happens. Many papers have analysed the approach to flat as a function of time but I will look at how the wobbling frequency depends on the angle of the disk to the horizontal.

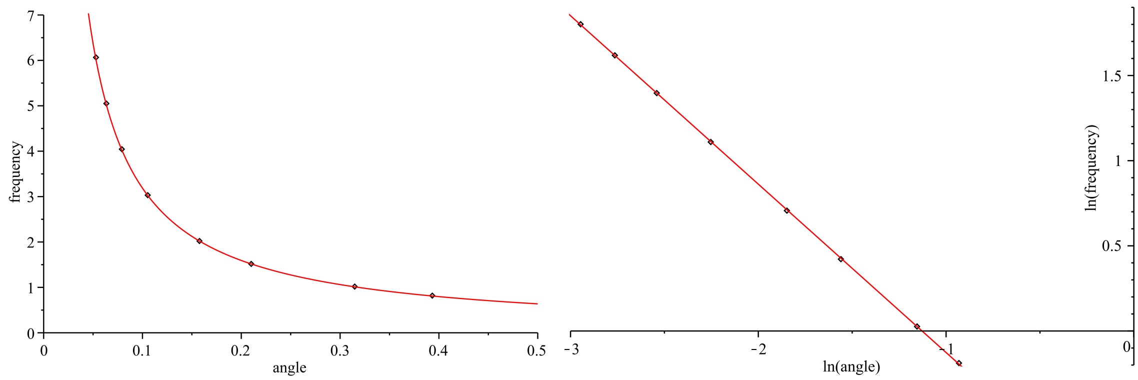
In reality the disk loses energy due to slipping and vibration which is why the disk eventually gets lower and finally stops. Here we will assume that any energy loss is taken from the potential energy only. As we mentioned earlier, the physical motion is approximately the regular motion when the potential energy remains constant so this makes sense. Therefore we analyse the approach to flat by solving the equations with the same initial kinetic energy but changing the initial angle. I define the wobbling frequency by the inverse of the time taken for the point of contact to draw out one closed circle under regular motion. Figure 8.1 shows the results of this exercise. They tell us that the wobbling frequency increases like the inverse of the angle. This agrees with what we observed physically as the disk's wobble frequency can be heard to approach a singularity as the angle of the disk to the horizontal tends to zero.



**(a)** No Penalty **(b)** $\epsilon = 0.3$ **(c)** $\epsilon = 0.1$

**(d)** $\epsilon = 0.05$ **(e)** $\epsilon = 0.01$ **(f)** Regular Motion

***Figure 7.4:*** *The path of the point of contact for the asymmetric disk with various levels of penalty. As demonstrated by (a) the behaviour of the asymmetric disk without the constraint of constant potential energy is extremely erratic. (b)-(e) show us how, with different levels of penalty, the constraint tries to force the disk to a more regular, more physical motion. The level of penalty required is much higher than for the symmetric disk to get motion which is even similar to pure regular motion.*




**(a)** **(b)**

***Figure 8.1:*** *As the angle of the symmetric disk is lowered, the wobbling frequency increases like* $\frac{1}{angle}$*, as shown in (a). (b) shows the* ln *graph of the same quantities to show that the relation gives a straight line.*

**Remark** Analysis of the wobbling frequency for the unbalanced disk would be much harder to determine. If the path of the point of contact is not periodic then how would we define one wobble of the disk? This is an interesting area for some further work.

□

# 9 Spinning Cylinder

The standard model for Euler's disk is for an infinitely flat disk. That is, we assume that the centre of mass is in the $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ plane and when the disk lies flat, the $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ plane coincides with the $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ plane in the spatial coordinates. Obviously a real disk has a physical height and the centre of mass will not be in the $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ plane as in the model.

This opens up the problem to a new class of physical objects: spinning cylinders. One such example of this is a frozen can of tuna. In fact, the frozen can of tuna gives a really good demonstration of the type of motion exhibited by Euler's disk. If you don't have an Euler's disk try putting a can of tuna in the freezer. The best are the cans with a rounded edge. It will give you an idea of how Euler's disk spins. It doesn't spin for $> 2$ minutes like Euler's disk but it is still interesting to play with.

Another key example is that empty pint glass which you knock in the bar and it starts rolling around on its edge. In this case the centre of mass is well out of the plane and the dynamics are very different. There is an analogy to having a broom balancing on your hand. If it is just a broom stick, it is very hard to balance. If the head of the broom is up in the air it makes it a lot easier to balance due to the higher inertia. This is a similar case with the glass. This is observed when sometimes you knock the glass and it will roll on its edge at a relatively slow rate for a surprisingly long time and it doesn't fall over. This happens when the initial velocity and spin of the glass is such that the glass is very close to falling over, but not quite there. The center of mass is almost above the point of contact and so the torque on the glass is low. This motion occurs with the spinning cylinder because the angle of the cylinder to the horizontal does not need to be to large to get the center of mass above the point of contact.

Luckily we have already formulated the equations for a centre of mass not in the plane in the equations for the flat disk with an off-centre COM. Recall that $\boldsymbol{\chi}$ is the vector from the centre of symmetry to the centre of mass. Instead of $\boldsymbol{\chi}$ being restricted to the $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ plane we can open it up to have an $\boldsymbol{E}_3$ component as well. We note that the centre of symmetry from which $\boldsymbol{\chi}$ originates is the centre of the circular base of the cylinder.

We will look at the simple case where the cylinder's mass distribution is entirely symmetric so that $\boldsymbol{\chi} = \boldsymbol{E}_3$. This should be a better model for the real Euler's disk. Consider a symmetric cylinder of height $2l$ so that the distance from the centre of the base to the COM is $l$.

Figure 9.1 shows the change in the path of the point of contact as we pull the center of mass out of the plane. We assume the cylinder is of uniform density and adjust the inertia tensor accordingly. The overall mass, radius ($r = 2$) and initial conditions all remain the same. The key thing to notice is that these diagrams are all drawn over the same time period $0 \leq t \leq 15$. We notice how, as the center of mass comes out of the plane, the number of rotations of the point of contact reduces. This is the effect I mentioned earlier with the pint glass. The cylinder can remain just balanced on its edge and roll slowly for a surprisingly long time since the center of mass is nearly above the point of contact. Here it is demonstrated by the numerics. If I increase the height of the center of mass to $l = 4$ then the cylinder falls over. The actual shape of the path travelled by the point of contact also changes slightly but this is not a significant effect. Figures 9.1a and 9.1b may look very different from the rest but this is simply because the motion here is approximately periodic. The path of the other diagrams are in fact the same overall shape but the phase is changing.

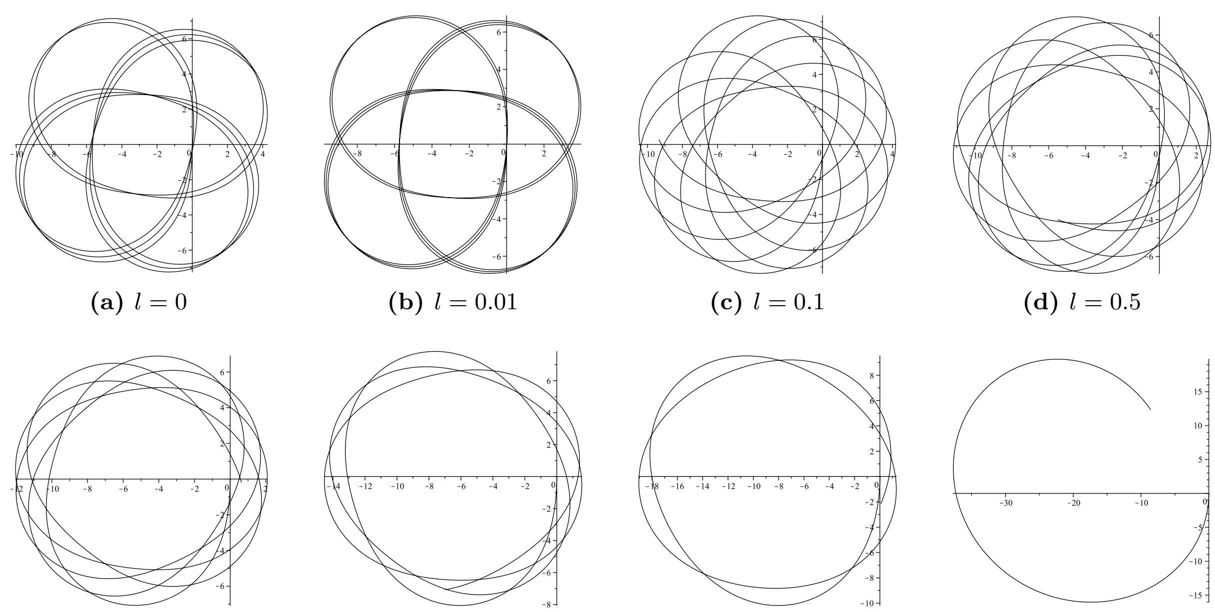
The motion of the disk isn't affected drastically for small $l$ but the change from $l = 0$ to $l = 0.1$ is enough to make us wonder if the flat disk model for Euler's disk is in fact that accurate. More work would need to be done in this area to determine the accuracy of the flat disk model to a nearly flat physical disk.

An interesting observation was that, under regular motion, the frequency of the wobble remained constant as we moved the center of mass out of the plane. We then realised that this regular motion of the disk is not a good model for a cylinder since it would never allow the cylinder to fall over! Even if we increase $l$ so the cylinder becomes very tall and thin, it draws out a perfect circle with the same wobbling frequency as if the cylinder were flat. Clearly non-physical, the cylinder should fall over.

## 10 Solving the Full System Numerically

For all the numerics we used Maple's `dsolve` command which uses *Fehlberg fourth-fifth order Runge-Kutta* method. Using MATLAB or C might have been faster, but the complexity of the equations was an obstacle to their conversion to a first order system for use in a general integrator. Maple has the ability to do this conversion for you so it seemed like the easiest way. Maple is able to convert the system to first order and then print it out for use in other applications. We did get Maple to do this for the symmetric case but the equations took up 3 pages and it seemed like a pointless exercise to try and program them into MATLAB or C. This is where the Euler angle formulation has its advantages: it is easier to decompose into a first order system and integrate numerically.

We were able to check that we were getting good accuracy by checking that the energy stayed constant. In most cases we were able to get 8 or more significant figures of accuracy so we are confident in the results.



(a) $l = 0$ (b) $l = 0.01$ (c) $l = 0.1$ (d) $l = 0.5$

(e) $l = 1$ (f) $l = 2$ (g) $l = 3$ (h) $l = 3.7$

***Figure 9.1:*** *Plots showing the variation in the position of the point of contact as we move the centre of mass out of the plane, for the same initial conditions and same time period. The number of rotations the point of contact make reduces as l increases.*

The unbalanced disk was hard to get decent numerical results for due to highly erratic behaviour, areas of very slow motion combined with areas of very fast motion, although these were clearly non-physical. The regular motion of the asymmetric disk took especially long to calculate. Maple's interface has quite a few bugs and we had many problems with Maple not being able to solve the problem one day but it could the next! *This is something I have experienced before with Maple. It is very temperamental.*

To deal with the erratic behaviour exhibited by the asymmetric disk, the key was to choose initial conditions that were as physical as possible. This involved quite a bit of trial and error but it worked out in the end and gave results which were much more meaningful.

For the 2D motion, the system was simple and we had no problems integrating and plotting any solutions to these problems. We have used Maple to make some simple animations of some of the 2D motions which can be seen on the attached CD of supplementary material. There are also animations which draw out the path of the point of contact for the asymmetric disk under regular motion. The CD also includes all of my Maple code as well as a PDF version of this document.

## 11 Conclusion

In this paper we have taken the well-known problem of the spinning symmetric disk and thrown it off balance. In reality the unbalanced disk exhibits different motions than the symmetric disk and we were able to show these with the mathematics. We invented a way to analyse the regular motion of the disk using this Geometric form of the equations. Previously this regular motion had only been analysed in Euler angles and only for a symmetric disk.

We included an overview of nonholonomic constraints. This was important to make the reader fully understand why we need to be extra cautious with the constraint and why standard Lagrangian mechanics methods (Hamilton's principle) cannot be used. We demonstrated what happens when we don't respect the constraint and obtain the wrong equations of motion. This is important for mathematicians since it is hard to believe what is said without some kind of evidence to back it up!

We were able to study extensively the 2D motions of the disk rolling on a given surface with some of the results turning out to be very elegant, something very satisfying to a mathematician. The equations also provided insight into motions which are not possible to analyse by simple observation. For example, the vertical disk rolling back and forth on a horizontal plane may be thought to be similar to a pendulum or simple harmonic motion. We were able to show this is not the case.

This project is by no means a complete analysis of the motions of the unbalanced disk. There are many areas where we would like to do some further work:

- Work on a symmetric ball rolling inside a sphere is a must and it would be excellent to prove (or disprove) that it is equivalent to the spherical pendulum.
- Further work could also be to analyse the phase of the disk. After a single wobble of the regular motion the disk will have rotated slightly. This represents a phase. Analysis of this for the symmetric case and then how it evolves as the disk becomes unbalanced would be very interesting.
- We would also like to find the equations of motion for two coupled rollers on an arbitrary surface. We could even expand this problem to $n$-rollers. It would be interesting to see if this same type of analysis could be used on other rolling bodies such as ellipsoids or egg shapes.
- One of the features of the asymmetric disk is that when it is vertical and spun very fast about the vertical axis, the center of mass rises and falls as it spins. In this situation the no-slip condition is not satisfied and we believe it to be a similar type of motion to the inverting of a spinning egg or a tippe top[6]. Some description of how/why this happens would be very satisfying.

This project has shown how geometric mechanics rather than more traditional methods can provide a more tractable solution to a complex problem. Given the power of this approach it would be interesting to see more numerical tools made available to solve the type of equations obtained via this method. This would provide an extremely powerful tool-kit for solving the problem of the unbalanced disk and other similar problems.

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